Lesson 35 Review 2

p. 503: 13. $y'' + 8y' + (\lambda+16)y = 0$ $\begin{cases} y(0)=0 \\ y(\pi)=0 \end{cases}$

$r^2 + 8r + (\lambda+16) = 0$

Roots $r = -4 \pm \sqrt{-\lambda}$

Case $\lambda = 0$: $r = -4, -4$ $\quad y = c_1 e^{-4x} + c_2 x e^{-4x}$

$y(0) = c_1 + c_2 \cdot 0 = c_1 \overset{\text{want}}{=} 0$ $\quad c_1 = 0$

So $y = c_2 x e^{-4x}$. $\quad y(\pi) = c_2 \pi e^{-4\pi} \overset{\text{want}}{=} 0$ $\quad c_2 = 0$.

Only $y \equiv 0$ sol^n! $\quad \lambda = 0$ is not an e-val.

Case $\lambda > 0$: $\lambda = \mu^2$ $\quad r = -4 \pm \mu i$

$y = c_1 e^{-4x} \cos \mu x + c_2 e^{-4x} \sin \mu x$

$y(0) = c_1 \overset{\text{want}}{=} 0$. $\quad \boxed{c_1 = 0}$

$y(\pi) = c_2 e^{-4\pi} \underbrace{\sin \mu\pi}_{\text{need} = 0} \overset{\text{want}}{=} 0$ $\quad$ Need $\mu\pi = n\pi$.

$\mu = n$, $n = 1, 2, 3, \ldots$ $\quad$ So e-vals are $\lambda = \mu^2 = n^2$.

For $\lambda = n^2$, get $y_n = e^{-4x} \sin nx$ $\quad$ take $c_2 = 1$.

Case $\lambda < 0$: $\lambda = -\mu^2$. $\quad r = -4 \pm \mu$

$$y = c_1 e^{(-4+\mu)x} + c_2 e^{(-4-\mu)x}$$

$$y(0) = c_1 + c_2 \overset{\text{want}}{=} 0 \quad \text{So} \quad \boxed{c_2 = -c_1}$$

$$\text{So} \quad y = c_1 e^{-4x}\left(e^{\mu x} - e^{-\mu x}\right)$$

$$y(\pi) = c_1 \underbrace{e^{-4\pi}}_{\neq 0} \underbrace{\Big(\underbrace{e^{\mu\pi}}_{>1} - \underbrace{e^{-\mu\pi}}_{<1}\Big)}_{>0} \overset{\text{want}}{=} 0$$

Must have $c_1 = 0$. Get only zero sol^n.

Fourier Integral:

$$f(x) \text{ "=" } \int_0^\infty A(w)\cos wx + B(w)\sin wx \, dw$$

where
$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos vw \, dv$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin vw \, dv$$

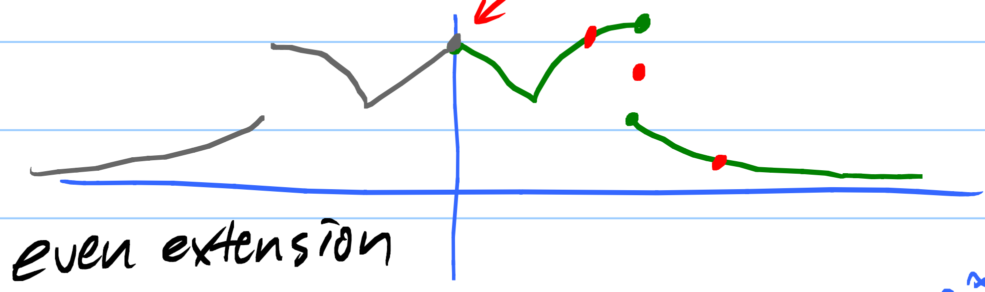
Note: $f(x)$ = Fourier Int at points x where f is continuous.

At jumps, Fourier Int. = midpoint of jump.

<u>Fourier Cosine Tranf</u>: $\tilde{\mathcal{F}}_c[f](w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$

$\tilde{\mathcal{F}}_c\left[\tilde{\mathcal{F}}_c[f]\right] = f$ ← true at $x = 0$

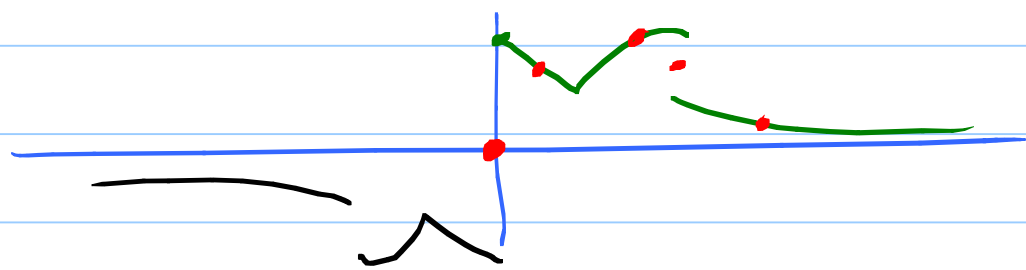
no jump



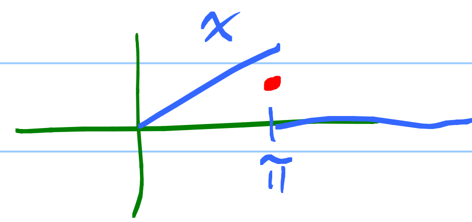
<u>Fourier Sine Transf</u>: $\tilde{\mathcal{F}}_s[f] = \sqrt{\frac{2}{\pi}} \int_0 f(x) \sin wx \, dx$

$\tilde{\mathcal{F}}_s\left[\tilde{\mathcal{F}}_s[f]\right] = f$ ← but $= 0$ at $x = 0$

$w = 0$
$\sin 0 = 0$



<u>EX</u>: Find $\tilde{\mathcal{F}}_c[f]$ where



$$f(x) = \begin{cases} x & 0 < x < \pi \\ 0 & x > \pi \end{cases}$$

$$\tilde{\mathcal{F}}_c[f] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi \underset{u}{x} \, \underbrace{\cos wx \, dx}_{dv}$$

$du = dx$

$v = \frac{1}{w} \sin wx$

$$= \sqrt{\frac{2}{\pi}} \left[x \left(\frac{1}{w} \sin wx \right) \Big|_{x=0}^{\pi} - \int_0^{\pi} \frac{1}{w} \sin wx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{w} \sin w\pi - 0 - \left[-\frac{1}{w^2} \cos wx \right]_{x=0}^{\pi} \right]$$

$$= \underbrace{\sqrt{\frac{2}{\pi}} \left[\frac{\pi \sin w\pi}{w} + \frac{\cos w\pi - 1}{w^2} \right]}_{H(w)}$$

$$\tilde{\mathcal{F}}_c [H(w)] = \begin{cases} x & 0 < x < \pi \\ \frac{\pi}{2} & x = \pi \\ 0 & x > \pi \end{cases}$$

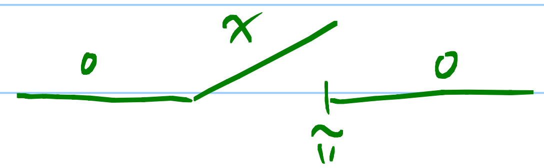
$$? = \int_0^{\infty} H(w) \cos \underset{\substack{\uparrow \\ x = \pi}}{\pi} w \, dw = \frac{1}{\sqrt{\frac{2}{\pi}}} \underbrace{\left(\begin{matrix} \text{Value of} \\ \text{"f" at} \\ x = \pi \end{matrix} \right)}_{\substack{\text{midpoint} \\ \text{of jump} = \\ = \frac{\pi}{2}}}$$

$$\tilde{\mathcal{F}}_c = \sqrt{\frac{2}{\pi}} \int$$

Note: No "Complex Fourier Series" on Exam 2.

The Fourier Transform (Complex Fourier Transf.)

$$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

f 

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\pi} x\, e^{-i\omega x}\, dx$$

$\int e^{cx}\,dx = \frac{1}{c}e^{cx}$ even if c is complex!

$$\int \underset{\substack{\uparrow \\ u}}{x}\,\underbrace{e^{cx}\,dx}_{dv} = \underline{\frac{x}{c}e^{cx} - \frac{1}{c^2}e^{cx}}$$

Take $c = -i\omega$.

$$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}}\left[\frac{x}{(-i\omega)}e^{-i\omega x} - \frac{1}{(-i\omega)^2}e^{-i\omega x}\right]_0^{\pi}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\pi e^{-i\omega\pi}}{-i\omega} - 0 - \left(\frac{1}{-\omega^2}e^{-i\omega\pi} - \left(\frac{-1}{\omega^2}\right)\right)\right]$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\left(\frac{i\pi e^{-i\pi\omega}}{\omega} + \frac{e^{-i\omega\pi}-1}{\omega^2}\right)}_{H(\omega)}$$

$$f(x) = \mathcal{F}^{-1}[H(\omega)]$$

$\mathcal{F}^{-1}$ has $i\omega x$ here.
$\mathcal{F}$ has $-i\omega x$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(\omega)\, e^{+i\omega x}\, d\omega$$

$\uparrow$ yes, but at $\tilde{\tau}$, $\int$ is $\frac{\pi}{2}$.

Prob: Laplace convolution of

$$\underset{f}{u(t-1)} \text{ and } \underset{g}{e^t}.$$

$$(f*g)(t) = \int_0^t f(\tilde{\tau})\, g(t-\tilde{\tau})\, d\tilde{\tau}$$

$$= \int_0^t \underbrace{u(\tilde{\tau}-1)}_{\substack{\text{on at} \\ \tilde{\tau}=1}} e^{t-\tilde{\tau}}\, d\tilde{\tau}$$

$$= \begin{cases} 0 & \text{if } t<1 \\ \int_1^t 1 \cdot e^{t-\tilde{\tau}}\, d\tilde{\tau} & \text{if } t>1 \end{cases}$$

$$\mathcal{L}[f*g] = \mathcal{L}[f] \cdot \mathcal{L}[g] = \frac{e^{-s}}{s} \cdot \frac{1}{s-1}$$