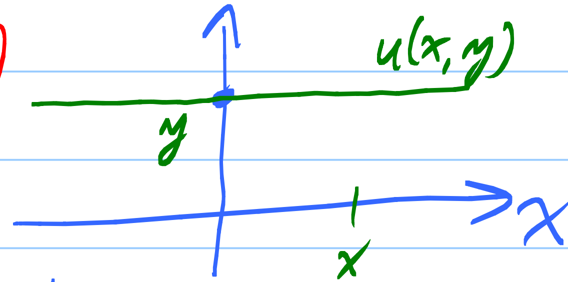
Lesson 36 PDE   36, 37, 38 due Wed. 11/20

Partial derivatives $\quad \frac{\partial u}{\partial x}(x,y) = \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h}$

Easiest PDE: $\quad \frac{\partial u}{\partial x} \equiv 0 \quad (*)$



See that u is constant on horizontal lines.

$u(x,y) = C(y) \leftarrow$ constant might depend on y.

Vice versa: If $u(x,y) = g(y) \leftarrow$ fcn of y alone

then $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} g(y) \equiv 0$. ✓

Sol^n to $(*)$ is $u(x,y) = C(y)$ where $C(y)$ is an arbitrary function of y.

Prob: Solve $\frac{\partial u}{\partial x} = e^y \sin 2x$.

$$u = \int e^y \sin 2x \, dx = e^y\left(-\tfrac{1}{2}\cos 2x\right) + C(y)$$

"partial integral w.r.t x, treat y like a const."

arb const might depend on y

Why it is right: $\frac{\partial}{\partial x}\Big[\underbrace{u - \left(-\tfrac{1}{2}e^{y}\cos 2x\right)}_{(*)}\Big] = 0$

So $(*)$ = fcn of y. ✓

EX: $u_{xx} + u = 0$ ← think: y is floating along as a parameter.

$$u = c_1(y)\cos x + c_2(y)\sin x$$

c_1 and c_2 need to arbitrary fcns of floating var.

First order linear ODE

$$A(x)\frac{dy}{dx} + B(x)y = C(x)$$

Step 1: Divide by $A(x)$:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Standard form!

Step 2: Find integrating factor: $\mu = e^{\int P(x)\,dx}$

Step 3: Mult. Stand. Form

PDE

$$\frac{1}{3}\frac{\partial u}{\partial x} + x^3 u = 0$$

1. $\frac{\partial u}{\partial x} + \underbrace{3x^3}_{P(x)}u = 0$

2. $\mu = e^{\int 3x^3\,dx} = e^{\frac{3}{4}x^4}$

3. $\underbrace{e^{\frac{3}{4}x^4}\left[\frac{\partial u}{\partial x} + 3x^3 u\right]} = 0$

ODE by μ:

$$\underbrace{\mu\left[\frac{dy}{dx}+Py\right]}=\mu Q$$

$$\frac{d}{dx}\left[\mu y\right]=\mu Q$$

Step 4: Integrate:

$$\mu y=\int \mu Q\,dx + C$$

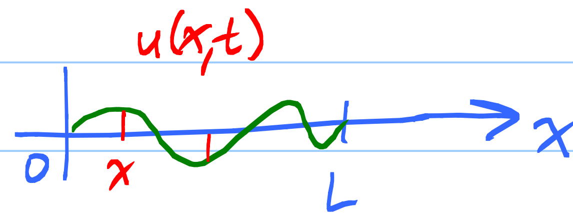
Step 5: Solve for y.

$$\frac{\partial}{\partial x}\left[e^{\frac{3}{4}x^4}u\right]=0$$

4. $e^{\frac{3}{4}x^4}u = C(y)$

5. $u = C(y)e^{-\frac{3}{4}x^4}$

Vibrating String Problem:



Wave Eqn. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Boundary Conditions (B.C.) $\begin{cases} u(0,t)=0 \\ u(L,t)=0 \end{cases}$

Intitial Conditions (I.C.)

$u(x,0)=f(x) \leftarrow$ Know shape at time 0

$\frac{\partial u}{\partial t}(x,0)=g(x) \leftarrow$ know intial velocity too

Separation of variables: $u(x,t) = X(x)T(t)$

PDE: $$\frac{\partial^2}{\partial t^2}[XT] = c^2 \frac{\partial^2}{\partial x^2}[XT]$$

$$X T'' = c^2 X'' T$$

"Separate variables:" $$\underbrace{\frac{T''}{c^2 T}}_{\text{fcn of } t} \overset{!}{=} \underbrace{\frac{X''}{X}}_{\text{fcn of } x} = \lambda$$

$\uparrow$ const.!

X problem: $X'' - \lambda X = 0$

B.C. translate: $X(0) = 0, \; X(L) = 0$

Sturm Liouville Prob.

why: $u(L,t) = \underbrace{X(L)}_{\text{need}=0} T(t) = 0$ ← want

don't want $T(t) \equiv 0$.

Got e-vals $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$, e-fcns $X_n(x) = \sin\frac{n\pi x}{L}$

T prob: $T'' - c^2 \lambda_n T = 0$

$$T = A_n \cos\frac{cn\pi}{L}t + B_n \sin\frac{cn\pi}{L}t$$

Get $u_n(x,t) = X_n(x) T_n(t)$ sol^{n}s.

Superposition principle:

Try $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

Why: PDE is linear.

B.C. are homogeneous

$$u(L,t) = \sum_{n=1}^{\infty} \underbrace{u_n(L,t)}_{0} = 0$$

Last thing to do: I.C.

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}\right)$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(-\frac{cn\pi}{L} A_n \sin\frac{cn\pi t}{L} + \frac{cn\pi}{L} B_n \cos\frac{cn\pi t}{L}\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$ want

Aha! Fourier Sine Series

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}$$

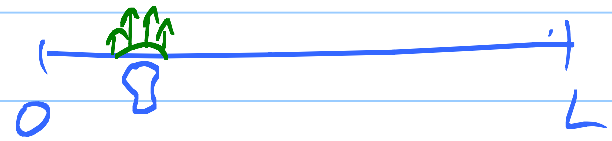
$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \left(\frac{cn\pi}{L} B_n\right) \sin\frac{n\pi x}{L} = g(x)$$ want

↑ $\frac{cn\pi}{L} B_n$ is the Fr. Sine Series coeff. for g.

$$\frac{cn\pi}{L} B_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$$\boxed{B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx}$$

Piano Prob:



$f(x)$ = little bump

$g(x)$ = little bump

Best place to whack string: Irrational multiple of L. Safe to do that near an end.

Next time: Use trig identities

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$$