Lessons 37, 38 Vib. String Prob. (36, 37, 38 due Wed.)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{I.C.} \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \quad \text{B.C.} \begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$$

Wave Eqn

Got $u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}\right)$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \leftarrow \text{F. Sine Series for } f$$

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

Case: $f(x)$ given, but $g(x) \equiv 0$. (Pluck)

Sol^n $u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L}$

$$\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x+ct) + \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x-ct)$$

Aha! $= \frac{1}{2} f^*(x+ct) + \frac{1}{2} f^*(x-ct)$

Really? $u_t = \frac{1}{2} f'(x+ct)(c) + \frac{1}{2} f'(x-ct)(-c)$

$$u_{tt} = \frac{1}{2} f''(x+ct)\,c^2 + \frac{1}{2} f''(x-ct)(-c)^2$$

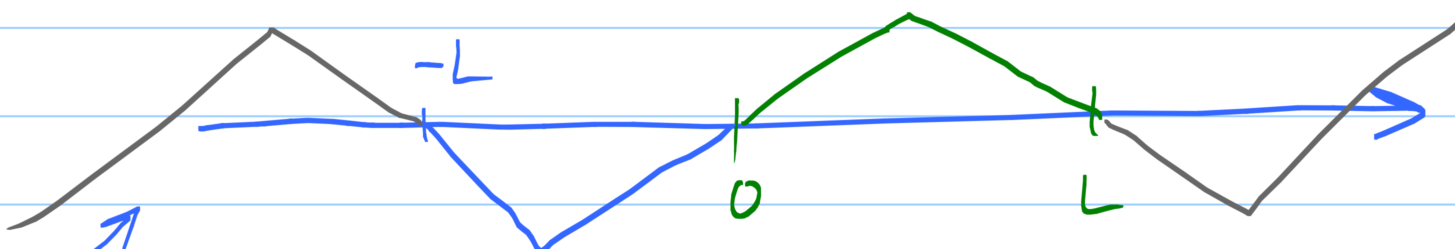
$$= c^2 u_{xx}$$ ✓

$$u(x,0) = \tfrac{1}{2} f(x) + \tfrac{1}{2} f(x) = f(x)$$ ✓

$$u_t(x,0) = \tfrac{1}{2} c f'(x) - \tfrac{1}{2} c f'(x) = 0$$ ✓

$$u(0,t) = \tfrac{1}{2} f(ct) + \tfrac{1}{2} f(-ct) \overset{?}{=} 0$$ Yes!

Important: f in the simple formula is the sum of a Fourier Sine Series of f on $[0, L]$, and hence, it is the odd periodic extension of f.



This is the f ($= f^*$ in the book) in simple formula.

General case where $g(x) \not\equiv 0$.

$$u(x,t) = \tfrac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \tfrac{1}{2c}\left[G(x+ct) - G(x-ct)\right]$$

where $G(x) = \int_0^x g(u)\,du$ where g here is the odd periodic extension of g on $[0,L]$.

<u>D'Alembert's sol^{n}:</u> $\quad c^2 u_{xx} = u_{tt}$

New variables: $\begin{cases} v = x + ct \\ w = x - ct \end{cases}$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = u_v \cdot 1 + u_w \cdot 1$$

$$u_{xx} = (u_v + u_w)_v \underbrace{\frac{\partial v}{\partial x}}_{1} + (u_v + u_w)_w \underbrace{\frac{\partial w}{\partial x}}_{1}$$

$$u_{xx} = u_{vv} + 2u_{wv} + u_{ww}$$

assuming u is C^2-smooth so that $u_{wv} = u_{vw}$

Get $\quad u_{tt} = c^2\left(u_{vv} - 2u_{wv} + u_{ww}\right)$.

Plug into $c^2 u_{xx} = u_{tt}$. Boils down to $u_{wv} = 0$.

$$\frac{\partial}{\partial w}\left[\frac{\partial u}{\partial v}\right] \equiv 0.$$

Baby PDE: $\quad \dfrac{\partial u}{\partial v} = \text{fcn of other var} = K(v)$

Baby PDE: $\quad u = \int K(v)\,dv + C(w)$

$$u = \varphi(v) + \psi(w)$$

$$u = \varphi(x+ct) + \psi(x-ct)$$

want

$$u(x,0) = \varphi(x) + \psi(x) = f(x)$$

want

$$u_t(x,0) = c\varphi'(x) - c\psi'(x) = g(x)$$

$$\varphi'(x) - \psi'(x) = \frac{1}{c} g(x)$$

(B) $$\varphi(x) - \psi(x) = \frac{1}{c}\int_0^x g(u)\,du$$

(A) $$\varphi(x) + \psi(x) = f(x)$$

Add: $$\varphi(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\int_0^x g(u)\,du\right)$$

(A) − (B) $$\psi(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\int_0^x g(u)\,du\right)$$

Get the same sol^n as above!