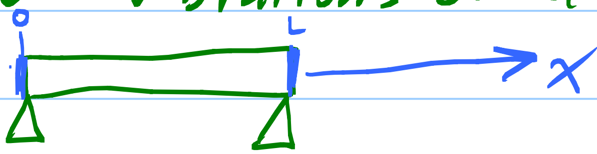
Lesson 39 (36, 37, 38 due Wed.)

Lesson 25 String problem

28, 29 } Heat problem

p. 551: 15, 16 Vibrations of a beam



$$\frac{\partial^2 y}{\partial t^2} = -c^2 \frac{\partial^4 y}{\partial x^4}$$

BC $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$ $\begin{cases} \frac{\partial^2 y}{\partial x^2}(0,t) = 0 \\ \frac{\partial^2 y}{\partial x^2}(L,t) = 0 \end{cases}$

Simply supported

not clamped, so zero curvature at ends

IC too.

15. Assume $u(x,t) = X(x)T(t)$. Separate vars.

$$\frac{X''''}{X} = \frac{T''}{-c^2 T} = \lambda, \text{ a const.}$$

<u>X problem</u>: $X^{(4)} - \lambda X = 0$

BC. $\begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases}$ $X''(0) = 0$, $X''(L) = 0$

Case $\lambda = 0$: $X^{(4)} \equiv 0$.

$$X(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

Question: Can we get non-zero sol^{n}s to ODE satisfying 4 BC?

Case $\lambda > 0$: ← only do this case for 15, 16.

$\lambda = \beta^4$ $(\beta > 0)$. $\quad X^{(4)} - \beta^4 X = 0$

$$r^4 - \beta^4 = 0 \leftarrow \text{Char. Eqn.}$$

$$(r^2 - \beta^2)(r^2 + \beta^2) = 0$$

$$(r-\beta)(r+\beta)(r^2+\beta^2) = 0 \qquad r = \pm\beta, \pm\beta i$$

$$X(x) = \underbrace{c_1 \cos\beta x + c_2 \sin\beta x}_{\pm\beta i} + \underbrace{c_3 e^{\beta x} + c_4 e^{-\beta x}}_{\pm\beta}$$

$$= A \cos\beta x + B \sin\beta x + \underbrace{C \cosh\beta x + D \sinh\beta x}_{\text{equiv spanning fcns.}}$$

Prob: Find values of β that allow non-zero sol^{n}s. (#16)

Case $\lambda < 0$: $\lambda = -\beta^4$ $(\beta > 0)$.

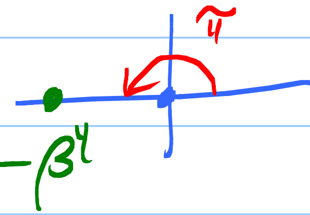
$$X^{(4)} + \beta^4 X = 0 \quad \text{plus BC.}$$

$$r^4 + \beta^4 = 0$$ ← 4 complex roots!

Looking for complex z such that $z^4 = -\beta^4$:

Key: Put complex numbers in polar form.

$$z = Re^{i\theta}, \quad -\beta^4 = \underline{\underline{\beta^4 e^{i\pi}}}$$



$$z^4 = \beta^4 e^{i\pi}$$

$$(Re^{i\theta})^4$$

$$R^4 e^{i4\theta} = \beta^4 e^{i\pi}$$ ← want

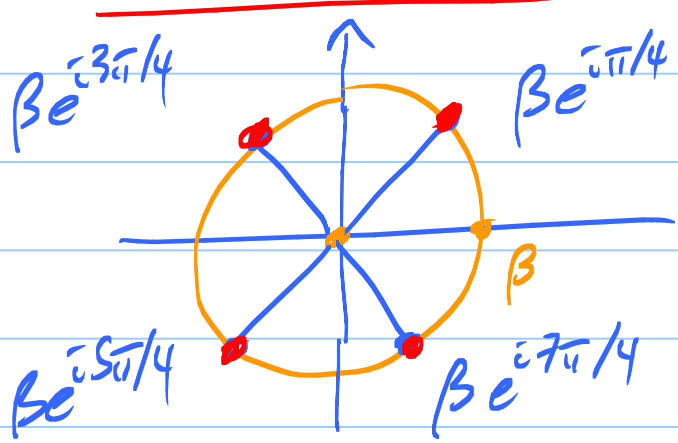
$$R^4 = \beta^4$$ ← radii same

$$\boxed{R = \beta}$$

$$4\theta = \pi + 2n\pi$$ ← angles must describe same spot

$n = 0, \pm 1, \pm 2, \ldots$

$$\boxed{\theta = \frac{\pi}{4} + n\frac{\pi}{2}}$$



4 roots:

$$\pm \frac{\beta}{\sqrt{2}} \pm i \frac{\beta}{\sqrt{2}}$$

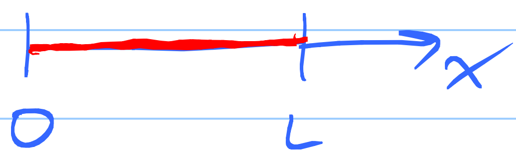
$$X(x) = c_1 e^{\beta x/\sqrt{2}} \cos\frac{\beta x}{\sqrt{2}} + c_2 e^{\beta x/\sqrt{2}} \sin\frac{\beta x}{\sqrt{2}}$$

$$+ c_3 e^{-\beta x/\sqrt{2}} \cos\frac{\beta x}{\sqrt{2}} + c_4 e^{-\beta x/\sqrt{2}} \sin\frac{\beta x}{\sqrt{2}}$$

Alternatively: $r^4 = -\beta^4$

$$r^2 = \pm \beta^2 i \nearrow \begin{cases} r^2 = \beta^2 i \to r = \pm\beta e^{i\pi/4} \\ r^2 = -\beta^2 i \to r = \pm\beta e^{-i\pi/4} \end{cases}$$

Heat Prob: Hot wire with insulated ends.



$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Insulated ends: Grad (Temp) = 0 at ends.

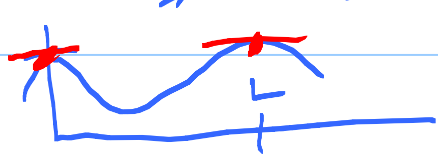
$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0.$$

Separate variables: $\dfrac{X''}{X} = \dfrac{T'}{c^2 T} = \lambda$

$$\boxed{\begin{array}{l} X'' - \lambda X = 0 \\ X'(0) = 0 \\ X'(L) = 0 \end{array}} \qquad \boxed{T' = c^2 \lambda T}$$

Case $\lambda = 0$: $X'' = 0$

$X(x) = c_1 x + c_2 \leftarrow$ line



Aha! $X(x) =$ horizontal line.

$$X(x) = c_2, \quad c_2 \neq 0.$$

$\lambda = 0$ is an <u>e-val</u>! Take $X_0(x) \equiv 1$ for the non-zero e-fcn.

<u>Case $\lambda > 0$</u>: $\lambda = \mu^2$ $(\mu > 0)$. $X'' - \mu^2 X = 0$

$$X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$r^2 - \mu^2 = 0$

$r = \pm \mu$

$$= A \cosh \mu x + B \sinh \mu x$$

$$X'(x) = \mu A \sinh \mu x + \mu B \cosh \mu x$$

want

$$X'(0) = \mu A \underbrace{\sinh 0}_{0} + \mu B \underbrace{\cosh 0}_{1} = \mu B = 0$$

$B = 0$.

So $X(x) = A \cosh \mu x$.

want

$$X'(L) = \mu A \sinh \mu L = 0 \quad \text{Oops!}$$

$\sinh \mu L \neq 0$.

So $A = 0$.

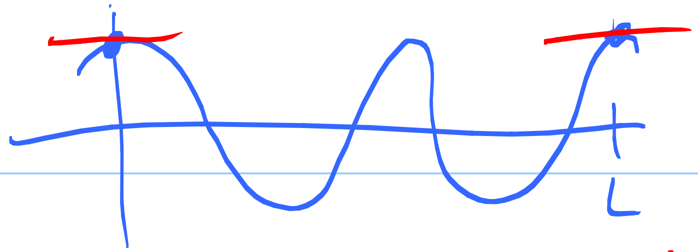
Get only the zero sol^n.

<u>Case $\lambda < 0$</u>: $\lambda = -\mu^2$ $(\mu > 0)$. $X'' + \mu^2 X = 0$

$$\underline{X(x) = c_1 \cos \mu x + c_2 \sin \mu x}$$

$r^2 + \mu^2 = 0$

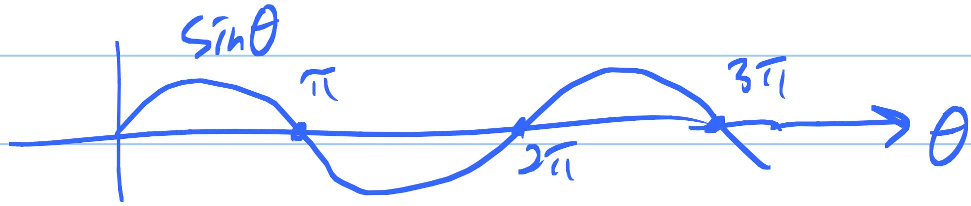
$r = \pm \mu i$



$X'(x) = -\mu c_1 \sin \mu x + \mu c_2 \cos \mu x$

$X'(0) = \mu c_2 = 0$ want. So $c_2 = 0$.

$X'(L) = -\mu c_1 \sin \mu L = 0$ want



$\mu L = n\pi$

$n = 1, 2, 3, \ldots$

$\mu = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

So $\lambda = -\mu^2 = -\left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$

Non-zero sol^n $X_n(x) = \cos \frac{n\pi x}{L}$

take $c_1 = 1$.

Sol^n: $u(x,t) = A_0 X_0(x) T_0(t) + \sum_{n=1}^{\infty} A_n X_n(x) T_n(t)$

T problem: $T' = \lambda c^2 T$

$\lambda = -\left(\frac{n\pi}{L}\right)^2$

$T_n(t) = e^{-(n\pi c/L)^2 t}$

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-(n\pi/L)^2 t}$$

Last thing: IC $u(x,0) = f(x)$

Aha! The A_n's are the Fourier Cosine Series.

want

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x)$$

Remark: Let $t \to \infty$.

$$\lim_{t\to\infty} u(x,t) = A_0$$

Ave Initial temp.