Lesson 40 12.6 (39, 40, 41 due Wed. after Thanksgiving)

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, \underbrace{e^{-\left(\frac{cn\pi}{L}\right)^2 t}}_{\to 0 \text{ as } t\to\infty}$$

want

Want $u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x)$

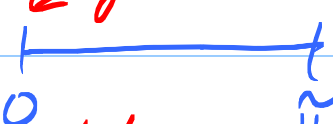
$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$\lim_{t\to\infty} u(x,t) = A_0 \leftarrow$ average initial temp!

Other boundary conditions:

#4 on Exam 2:

$y'(0)=0$ $y(\pi)=0$



insulated end at $x=0$

iced end at $x=\pi$.

Separate variables.

$X'' - \lambda X = 0$

e-vals: $\lambda_n = -\left(\frac{1}{2}+n\right)^2$ $n = 0, 1, 2, \ldots$

e-fcns: $X_n(x) = \cos\left(\frac{1}{2}+n\right)x$

Sturm-Liouville Prob. X_n are a complete orthogonal set.

Get sol^n $u(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{1}{2}+n\right)x\, e^{-\left(\frac{1}{2}+n\right)^2 c^2 t^2}$

Last thing: Want $u(x,0) = \sum_{n=1}^{\infty} c_n \cos\left(\tfrac{1}{2}+n\right)x \overset{\text{want}}{=} f(x)$

S-L Theory says expansion exists.

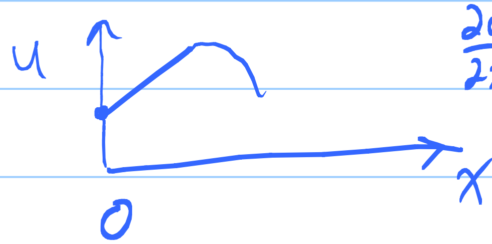
Multiply by $\cos\left(\tfrac{1}{2}+m\right)x$:

$$\int_0^{\pi} f(x)\cos\left(\tfrac{1}{2}+m\right)x\,dx = \sum_{n=1}^{\infty} c_n \int_0^{\pi} \cos\left(\tfrac{1}{2}+\underset{\substack{\uparrow\\ n}}{n}\right)x \cos\left(\tfrac{1}{2}+\underset{\substack{\uparrow\\ m}}{m}\right)x\,dx$$

$$= c_m \underbrace{\int_0^{\pi} \cos^2\left(\tfrac{1}{2}+m\right)x\,dx}_{>0}$$

Generalized Fourier Series!

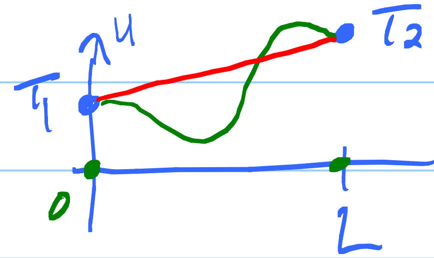
Radiative end: (Temp gradient) $= \pm K$ (Temp)



$$\frac{\partial u}{\partial x}(0,t) = -K\,u(0,t)$$

$$X'(0) + K X(0) = 0$$

What if we want:



Ouch! We needed $T_1 = 0$ and $T_2 = 0$ so we could add up sol^{n}s and not mess up boundary conditions.

Classic trick: What is the steady state solution?

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\downarrow \; t \to \infty$$

$$0 = c^2 \frac{d^2}{dx^2} u$$

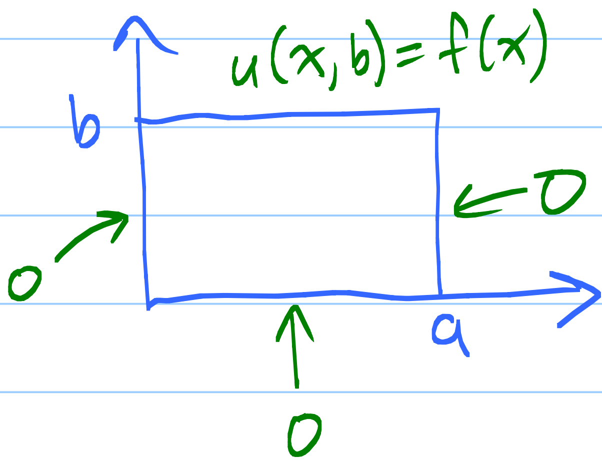
$$u = c_1 x + c_2$$

Get $u = T_1 + \frac{x}{L}(T_2 - T_1). = u_\infty(x)$

Big Idea: Let $U(x,t) = \underbrace{u(x,t)}_{\text{want}} - u_\infty(x)$

Aha! U satisfies our favorite heat prob with homog B.C.'s.

Hot rectangular plate problem:



$$\frac{\partial u}{\partial t} = c^2 \Delta u$$

$$\downarrow \leftarrow \text{steady state}$$

$$0 = c^2 \Delta u$$

$$u(x,y) = X(x)\, Y(y)$$

PDE: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda, \quad \text{const.}$$

BC?

Bottom: $u(x,0) = X(x)\underbrace{Y(0)}_{=0} = 0$ want

Left: $u(0,y) = \underbrace{X(0)}_{=0}Y(y) = 0$ want

Right: $u(a,y) = \underbrace{X(a)}_{=0}Y(y) = 0$ want

$$\boxed{\begin{aligned} &X'' - \lambda X = 0 \\ &X(0) = 0, \; X(a) = 0 \end{aligned}}$$

easy

$$\left[\begin{aligned} &Y'' + \lambda Y = 0 \\ &Y(0) = 0 \end{aligned}\right.$$

$$\lambda = -\left(\frac{n\pi}{a}\right)^2$$

$$X_n(x) = \sin\frac{n\pi x}{a}$$

$$Y'' - \left(\frac{n\pi}{a}\right)^2 Y$$

$$r^2 - \left(\frac{n\pi}{a}\right)^2 = 0$$

$$r = \pm\frac{n\pi}{a}$$

$$Y(y) = c_1 \cosh \frac{n\pi y}{a} + c_2 \sinh \frac{n\pi y}{a}$$

want

Need $Y(0) = c_1 = 0$

Take $c_2 = 1$, Get $Y_n(y) = \sinh \frac{n\pi y}{a}$

Solⁿ $u(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$