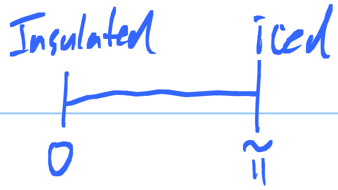
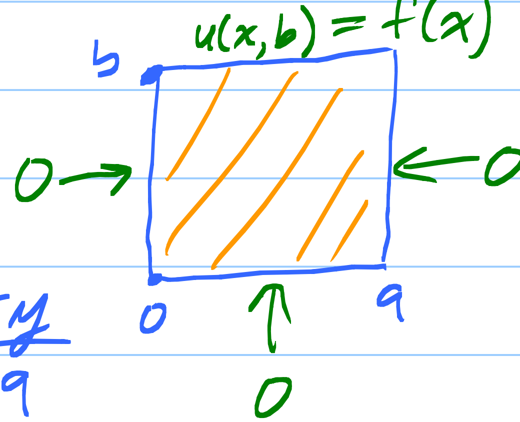
Lesson 40, 41 (39, 40, 41 due Wed. after break)



Orthogonal fcns $\cos(\frac{1}{2}+n)x$ not $\sin$!
fixed on Aftermath.



Laplace Eqn: $\Delta u = 0$.

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

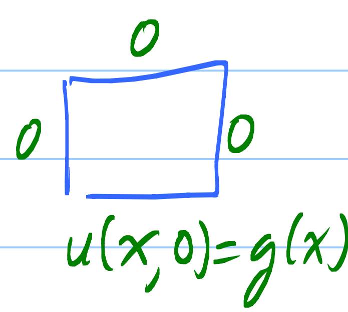
Last thing: BC at top.

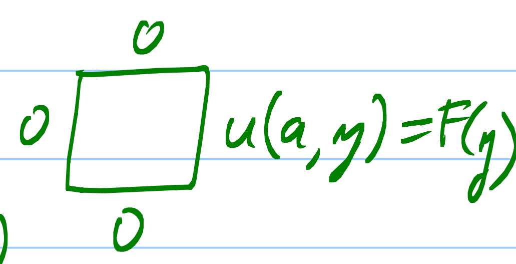
$$u(x,b) = \sum_{n=1}^{\infty} \underbrace{\left(A_n \sinh\frac{n\pi b}{a}\right)}_{\text{Aha! Fourier Sine Series coeff for } f} \sin\frac{n\pi x}{a} \overset{\text{want}}{=} f(x)$$

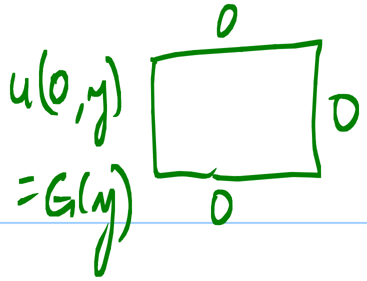
$$A_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$$

Solve for A_n. Good news: $\frac{1}{\sinh\frac{n\pi b}{a}} \to 0$ very fast as $n \to \infty$.

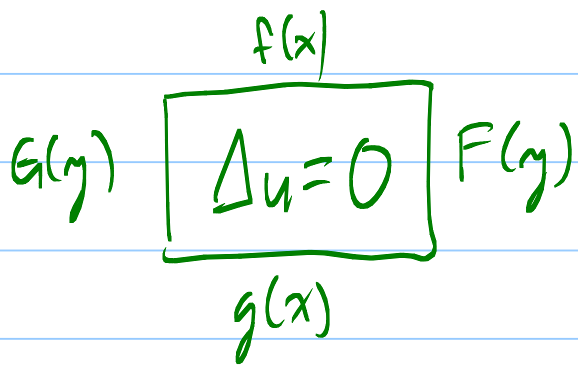
Do 3 more problems







Add up all 4 sol'ns to solve the Dirichlet Problem for the rectangle:



Temp given on edges.

12.6 done.

12.10 Laplacian in polar coord.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

p. 591: 4abc.

a) Apply Δ in polar coords to

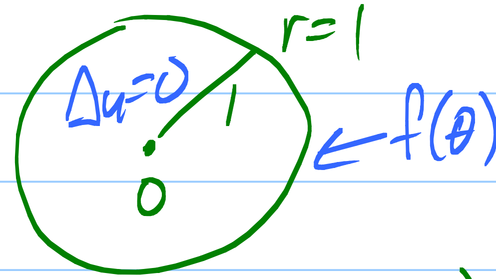
$r^n \cos n\theta$ and $r^n \sin n\theta$.

Show their Δ is $= 0$.

b) (*) $u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n r^n \cos n\theta + b_n r^n \sin n\theta\right)$

also satisfies $\Delta u = 0$ on disc

(if it is justified to put derivatives under Σ).



c) Aha!

$$u(1,\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin \theta)$$

is just a Fourier series. If I want to solve the Dirichet Prob. on the unit disc, $\Delta u = 0$ inside, $u(1,\theta) = f(\theta)$ given ($f(0) = f(2\pi)$), then (*) does it.

Lesson 41, 12.7 Heat problems on long wires.

Using Fourier Transform, $\hat{f}$, $\tilde{\mathcal{F}}_c$, $\tilde{\mathcal{F}}_s$ to solve heat problems (insulated end, iced end, no end). See Lessons 31, 32.

Hot infinite wire, given initial temp.

$u(x,t) = \text{temp}$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{plus Initial Cond, } u(x,0) = f(x).$$

New approach: Separate variables. Look for sol^{n}s $u(x,t) = X(x)T(t)$. Get

$$\frac{X''}{X} = \frac{T'}{c^2 T} = \lambda, \text{ const.}$$

No conditions on λ! Get many more sol^{n}s than when we had BCs.

Look at wiggling sol^{n}s. Let $\lambda = -w^2$. $(\lambda \le 0)$.

$X'' + w^2 X = 0$ $\qquad T' = -c^2 w^2 T$

$X(x) = A\cos wx + B\sin wx$ $\qquad T = Ke^{-c^2w^2t}$

Get sol^n $u(x,t) = (A\cos wx + B\sin wx)e^{-c^2w^2t}$

Big idea. No restriction on w.

Hmmm. Riemann sum is a linear combo of sol^{n}s.

$$\sum_{n=1}^{N}\big(\underbrace{A(w_j)}_{\text{coeff}}\cos w_j x + \underbrace{B(w_j)}_{\text{coeff}}\sin w_j x\big)e^{-c^2w_j^2t}\,\Delta w_j$$

$$\downarrow$$

$$(*) \qquad u(x,t) = \int_0^\infty \big(A(w)\cos wx + B(w)\sin wx\big)e^{-c^2w^2t}\,dw$$

If all goes well, this will solve PDE and we can diff under $\int$ to see that. Can solve

IC $\quad u(x,0) \overset{\text{want}}{=} f(x) = \int_0^\infty A(w) \cos wx + B(x) \sin wx \, dw$

Aha! This is just the Fourier Integral for f:

$$A(w) = \frac{1}{\tilde{\pi}} \int_{-\infty}^{\infty} f(v) \cos wv \, dv$$

$$B(w) = \frac{1}{\tilde{\pi}} \int_{-\infty}^{\infty} f(v) \sin wv \, dv$$

p. 574: 2-5. Just find $A(w)$ and $B(w)$ for given f and plug it into (*)

2. $f(x) = 1$ if $-a < x < a$ and 0 otherwise.

$$A(w) = \frac{1}{\tilde{\pi}} \int_{-a}^{a} 1 \cos wv \, dv = \frac{1}{\tilde{\pi}} \left[\frac{1}{w} \sin wv \right]_{v=-a}^{a}$$

$$B(w) = \frac{1}{\tilde{\pi}} \int_{-a}^{a} \underbrace{\underset{\text{even}}{1} \cdot \underset{\text{odd}}{\sin wv}}_{\text{odd}} \, dv = 0$$

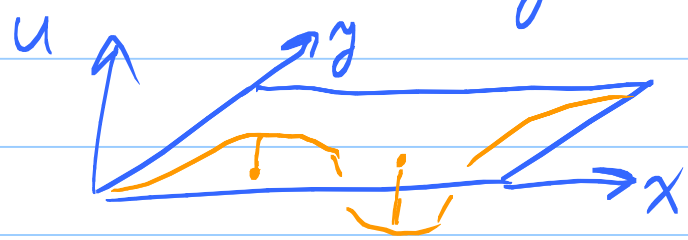
over a symmetric interval about 0

Remark: Can do calculus to clean up (*):

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v) \exp\left(\frac{-(x-v)^2}{4c^2 t} \right) dv$$

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x+2cz\sqrt{t})\, e^{-z^2}\, dz$$

Lesson 42 12.9 Vibrating square membrane.



$u(x,y,t)$ = displacemnt of drum at (x,y) at time t

Physics: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \Delta u$ PDE

BC $u=0$ on edges

IC initial shape and speed of drum head.

Separate variables: $u(x,y,t) = X(x)Y(y)T(t)$