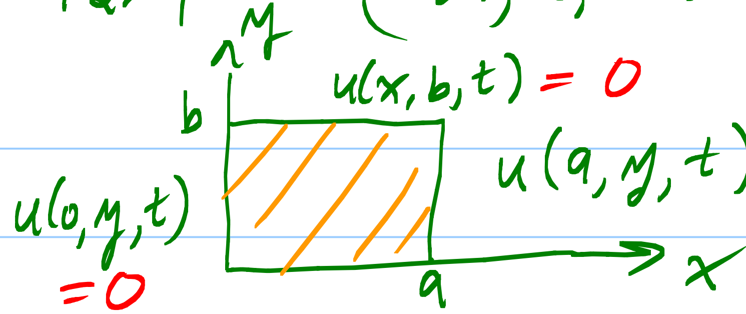
Lesson 42: 12.9 (39, 40, 41 due Wed after TG)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$



$$u(x, 0, t) = 0$$

IC $u(x, y, 0) = f(x, y) \leftarrow$ given

$$\frac{\partial u}{\partial t}(x, y, 0) = g(x, y) \leftarrow \text{given}$$

Sol^n: Try $u(x, y, t) = X(x)\, Y(y)\, T(t)$

want

BC: $u(a, y, t) = \underbrace{X(a)}_{=0}\, Y(y)\, T(t) = 0$

Need $X(a) = 0$.

Repeat: $\left[\begin{array}{l} X(0) = 0,\ X(a) = 0 \\ Y(0) = 0,\ Y(b) = 0 \end{array}\right]$

PDE: $$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$X Y T'' = c^2 \left(X'' Y T + X Y'' T \right)$$

Divide by XYT and c^2:

$$\frac{T''}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y} = \mu, \text{ a const.}$$

T-prob: $T'' - c^2 \mu T = 0$, no BC.

Easy. Save it.

Let $a = b = \pi$, $c = 1$, $g(x,y) \equiv 0$

X-prob: $\dfrac{X''}{X} + \dfrac{Y''}{Y} = \mu$

$$\frac{X''}{X} = \mu - \frac{Y''}{Y} = \lambda, \text{ a const.}$$

$$\left[\begin{array}{l} X'' - \lambda X = 0 \\ X(0) = 0,\ X(\pi) = 0 \end{array}\right] \quad \lambda_n = -n^2,\ X_n(x) = \sin nx$$

Y-prob: $\mu - \dfrac{Y''}{Y} = -n^2$

$$\left[\begin{array}{l} Y'' - (\mu + n^2) Y = 0 \\ Y(0) = 0,\ Y(\pi) = 0 \end{array}\right]$$

Only get non-zero sol^n if $\mu + n^2 = -m^2$

Get non-zero sol^n $Y_m(y) = \sin my$.

Note: $\mu = -n^2 - m^2$

T-prob: $T'' + (n^2 + m^2) T = 0$

$$T_{nm} = c_1 \cos\sqrt{n^2+m^2}\,t + c_2 \sin\sqrt{n^2+m^2}\,t$$

Get $u_{nm}(x,y,t) =$

$$\sin nx \sin my \left(A_{nm} \cos\sqrt{n^2+m^2}\,t + B_{nm} \sin\sqrt{n^2+m^2}\,t \right)$$

Satisfies linear PDE and homog BC.

Finally, want IC to hold.

Try $u(x,y,t) = \sum_{n,m=1}^{\infty} u_{nm}(x,y,t)$

Note: Get $\frac{\partial u}{\partial t}(x,y,0) = 0$ by taking all B_{nm}'s $= 0$, so

$$u(x,y,t) = \sum_{n,m=1}^{\infty} A_{nm} \sin nx \sin my \cos\sqrt{n^2+m^2}\,t$$

want

Want $u(x,y,0) = \sum_{n,m=1}^{\infty} A_{nm} \sin nx \sin my = f(x,y)$

Double Fourier Series!

$\varphi_{nm}(x,y) = \sin nx \sin my$ are orthogonal on $[0,\pi]\times[0,\pi]$ with inner product

$$\langle \varphi, \psi \rangle = \iint_R \varphi\psi \, \underbrace{dA}_{\text{area}} =$$

$$= \int_0^\pi \int_0^\pi \varphi(x,y)\psi(x,y)\,dx\,dy$$

$$f = \sum A_{nm}\varphi_{nm}$$ ← multiply by φ_{NM}

$$f\varphi_{NM} = \sum A_{nm}\varphi_{nm}\varphi_{NM}$$

$$\iint f\varphi_{NM}\,dA = \sum A_{nm}\underbrace{\iint \varphi_{nm}\varphi_{NM}\,dA}_{\text{all } = 0 \text{ except } n=N,\, m=M \text{ one}}$$

$$= A_{NM}\cdot\frac{\pi^2}{4}$$

Get $A_{NM} = \dfrac{4}{\pi^2}\displaystyle\int_0^\pi\int_0^\pi f(x,y)\sin Nx \sin My\,dx\,dy$