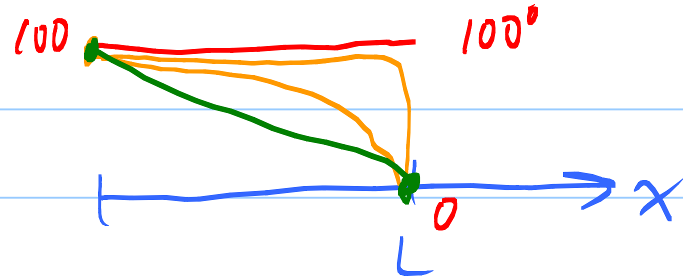
Lesson 43 12.10 (39, 40, 41 due Wed.)

p. 566: 10.



$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 100$$

$$u(L,t) = 0$$

$$u(x,0) = 100$$

$$u_\infty(x) = 100 - \frac{100}{L} x$$

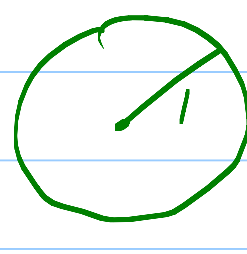
Trick: Let $U(x,t) = u(x,t) - u_\infty(x)$

U satisfies heat eqn. $\begin{cases} U(0,t) = 0 \\ U(L,t) = 0 \end{cases}$ BC

$$U(x,0) = u(x,0) - u_\infty(x) = 100 - \left(100 - \frac{100}{L} x\right)$$

$$= \frac{100}{L} x$$ ← IC

Vibrating circular drum problem:



$c = 1$

$$\frac{\partial^2 u}{\partial t^2} = \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

BC: $u(1,\theta,t) = 0$

IC: $u(r,\theta,0) = f(r,\theta)$ ← given

$\frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta)$ ↙

Separation of variables. Try $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$.

Hidden condition: $\Theta(0) = \Theta(2\pi)$, $\Theta'(0) = \Theta'(2\pi)$ } periodic bndry cond.

Θ problem: $\Theta'' + \lambda\Theta = 0$ with periodic BC

Sol^n is general Fourier Series!

PDE $$u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$$R\Theta T'' = R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta'' T$$

Divide by $R\Theta T$:

$$\frac{T''}{T} = \frac{R'' + \frac{1}{r}R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \mu,$$ a const.

For now, assume u does not depend on θ.
So $\Theta'' = 0$.

R problem: $R'' + \frac{1}{r}R' - \mu R = 0$

Multiply by r^2: $r^2R'' + rR' - \mu r^2 R = 0$

Bessel's eqn! with $R(1) = 0$ BC

Case $\mu = 0$: $r^2R'' + rR' = 0$, $R(1) = 0$

Euler Eqn: $ax^2y'' + bxy' + cy = 0$

Try $y = x^p$, $y' = px^{p-1}$, $y'' = p(p-1)x^{p-2}$

$$[\underbrace{ap(p-1) + bp + c}_{\text{need } = 0}]\, x^p = 0$$

Get roots p_1, p_2.

Gen^l Sol^n: $y = c_1 x^{p_1} + c_2 x^{p_2}$

Double root: p_1, p_1. Get $y = c_1 x^{p_1} + c_2 x^{p_1} \ln x$

Complex roots: $p = a \pm bi$ Get complex sol^n

$$x^{a+bi} = x^a x^{bi} = x^a e^{bi \ln x}$$

$$= \underbrace{x^a \cos(b \ln x)} + i\, \underbrace{x^a \sin(b \ln x)}$$

Two lin. ind. real sol^{n}s

Our Euler Eqn: $r^2 R'' + r R' = 0$

$$p(p-1) + p = 0$$

$$p^2 = 0$$

$p = 0$ is a double root.

$$R(r) = c_1 r^0 + c_2 r^0 \ln r$$

$$= c_1 + c_2 \underbrace{\ln r}_{}$$

Ouch! Drum ripper

c_2 must be zero!

So $R(r) = c_1$. Also $R(1) = 0$.

Ouch! Must have $c_1 = 0$ too. Get no non-zero sol^{n}s for $\mu = 0$.

<u>Case $\mu > 0$:</u> Can show there are no non-zero bounded sol^{n}s in this case.

<u>Case $\mu < 0$:</u> Write $\mu = -k^2$.

$$r^2 R'' + r R' + k^2 r^2 R = 0$$

$$\text{with } R(1) = 0$$

To put in Standard Form, change variables $s = kr$. Use chain rule and plug into ODE:

$$(*) \quad s^2 \frac{d^2 R}{ds^2} + s \frac{dR}{ds} + s^2 R = 0$$

Besel's eqn of order 0

<u>How to find a sol^n:</u> Try a power series sol^n

$$R(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots$$

$$R'(s) = a_1 + 2a_2 s + 3a_3 s^2 + \cdots$$

$$R''(s) = 2a_2 + 3\cdot 2a_3 s + \cdots$$

Plug into (*) and collect coeff of s^n.

Set them all $= 0$. Do "method of undet coeff" forever, $n = 0, 1, 2, 3, \ldots$.

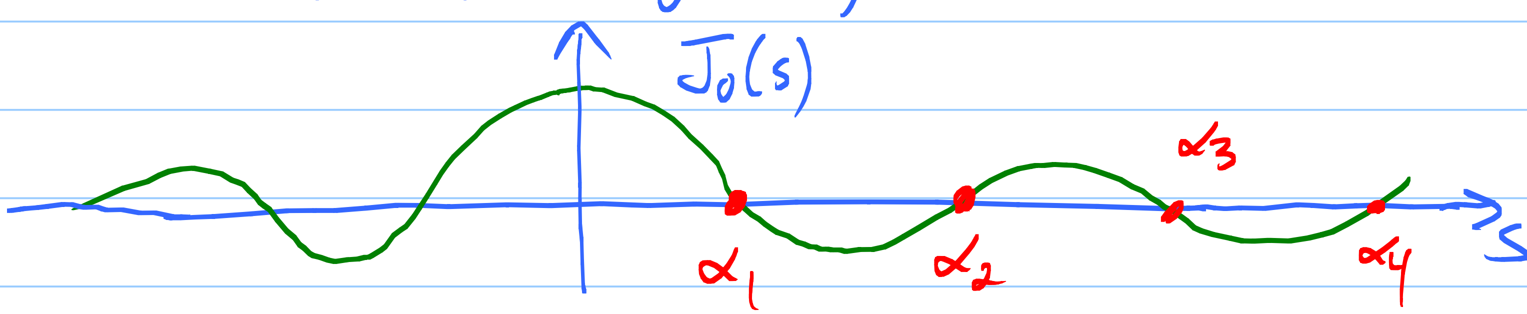
Get Bessel Function of order zero

$J_0(s)$ = power series with $(n!)^2$ in denominators!

Soln: $R(s) = c_1 J_0(s) + c_2 \underbrace{Y_0(s)}_{\text{blows up at } s=0! \ (s=kr)}$

c_2 must $= 0$.

Get $R(r) = J_0(kr)$



Still need $R(1) = 0$, i.e., $k \cdot 1 = \alpha_n$

E-vals: $\mu = -k^2 = -\alpha_n^2$

E-fcns: $R_n(r) = J_0(\alpha_n r) \quad n = 1, 2, 3, \ldots$

Sturm-Liouville: $R_n(r)$ are orthogonal on $[0,1]$.

T prob: $T'' - \mu T = 0$

$\mu = -\alpha_n^2$

$$T_n(t) = A_n \cos \alpha_n t + B_n \sin \alpha_n t$$

Get $u_n(r,t) = A_n J_0(\alpha_n r) \cos \alpha_n t + B_n J_0(\alpha_n r) \sin \alpha_n t$

$u(r,0) = f(r)$ $\qquad$ $\frac{\partial u}{\partial t}(r,0) = g(r)$

Take $g \equiv 0$ for now.

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \cos \alpha_n t$$

Last thing: $u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) = f(r)$ want

Multiply by $J_0(\alpha_m r)$ and integrate 0 to 1 to pin down A_m.