Last lecture.

Final Exam: Wed., Dec. 11, 1-3 pm, ME 1061

Dim Row Space = Dim Col Space

$$\underbrace{\text{Nullity}(A)}_{\#\text{free vars}} + \underbrace{\text{Rank}(A)}_{\#\text{bound vars}} = \#\text{cols } A \leftarrow \#\text{vars}$$

$$\text{Nullity}(A) = \dim \text{null}(A) = \dim\{\vec{x} : A\vec{x} = \vec{0}\}$$

On campus: #2 pencil

Off campus: something dark.

2 crib sheets handwritten on both sides.

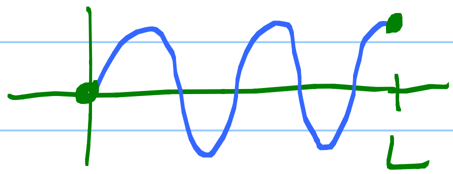
Laplace Transform Table = page 1, peel off.

Fourier Series $[-L, L]$

Fourier Sine, Cosine Series $[0, L]$

No complex Fourier <u>Series</u>, but yes for <u>Transform</u>.

Sturm-Liouville probs: Don't need to write down formulas. Just need to know how to do this kind of prob: Find all positive e-vals for S-L prob $X'' + \lambda X = 0$ with $X(0) = 0$, $X'(L) = 0$.



$\sin \mu x$

Fourier Integral: $f(x) = \int_0^\infty A(w)\cos wx + B(w)\sin wx \, dw$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty}$ etc., $B(w) = \ldots$

Sol^n to heat prob on $\mathbb{R}$: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ on $\mathbb{R}$ with $u(x,0) = f(x)$.

$$u(x,t) = \int_0^{\infty} \left(A(w)\cos wx + B(w)\sin wx\right) e^{-c^2 w^2 t}\, dw$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(p)\cos pw\, dp$, etc.

See prob. 27.

Fourier Sine, Cosine Transforms $\left(\sqrt{\frac{2}{\pi}}\,!\right)$

$\tilde{\mathcal{F}}_c$, $\tilde{\mathcal{F}}_s$

<u>The</u> Fourier Transform $\tilde{\mathcal{F}}[f] = \hat{f}$

$\tilde{\mathcal{F}}_c[f']$, $\tilde{\mathcal{F}}_c[f'']$, same for $\tilde{\mathcal{F}}_s$, $\tilde{\mathcal{F}}$.

<u>PDE</u>: Wave eqn (String)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\left.\begin{array}{l} u(0,t)=0 \\ u(L,t)=0 \end{array}\right\} \text{BC}$$

$$\left.\begin{array}{l} u(x,0)=f(x) \\ \frac{\partial u}{\partial t}(x,0)=g(x) \end{array}\right\} \text{IC}$$

Two ways: Sep. Var., Fourier Sine Series

D'Alembert's sol^n: Take <u>odd</u> periodic extension of f and g.

$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

$$+ \frac{1}{2c}\left[G(x+ct) - G(x-ct)\right]$$

where $G(x) = \int_0^x g(u)\,du$.

Prob. 26. $u_{tt} = 4u_{xx}$ on $\mathbb{R}$ $\quad u(x,0) = \sin x$

$c=2$ $\quad \frac{\partial u}{\partial t}(x,0) = \cos x$

$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

$$\frac{\partial u}{\partial t}(x,t) = c\varphi'(x+ct) - c\psi'(x-ct)$$

want

$$u(x,0) = \varphi(x) + \psi(x) = \sin x \qquad (A)$$

$$\frac{\partial u}{\partial t}(x,0) = 2\varphi'(x) - 2\psi'(x) = \cos x$$

$$2\varphi(x) - 2\psi(x) = \sin x$$

$$\varphi(x) - \psi(x) = \frac{1}{2}\sin x \qquad (B)$$

Cramer's Rule, or HS: $\quad \varphi(x) = \frac{3}{4}\sin x$

$\psi(x) = \frac{1}{4}\sin x$

Heat Eqn: ∞ long. Fourier Integral method.

Hot wire: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\left.\begin{array}{l} u(0,t)=0 \\ u(L,t)=0 \end{array}\right\}$ BC iced ends (homog BC)

IC: $u(x,0) = f(x)$

Crib sheet: Formula for sol^n. (Fourier Sine Series)

Sum of $(\sin \mu x)\, e^{-c^2\mu^2 t}$

Other problem: Insulated ends: BC $\left\{\begin{array}{l} \frac{\partial}{\partial x} u(0,t)=0 \\ \frac{\partial u}{\partial x}(L,t)=0 \end{array}\right.$

Get Fourier Cosine series.

Keep straight: Convolution for $\mathcal{L}$ ← know
Convolution for $\tilde{\mathcal{F}}$ ← not on exam

Note: Bessel functions are $\perp$ with respect to weight function r.
Higher order Bessel functions arise when θ comes in.

Legendre's eqn and polynomials come up in 3-d wave eqn (vibrating ball), spherical coords. L-eqn in radial direction.

Fourier series are $\perp$ expansions.

$$f(x) = c_1 + c_2 x^7 + c_3 \cos x$$

1) show that $1, x^7, \cos x$ are $\perp$ on $[-\pi, \pi]$

$$(1, x^7) = \int_{-\pi}^{\pi} \underbrace{1 \cdot x^7}_{\text{odd}} \, dx = 0 \quad \checkmark$$

$$(1, \cos x) = \int_{-\pi}^{\pi} \underbrace{\cos x}_{\text{even}} \, dx = 2\int_0^{\pi} \cos x \, dx = 0$$

by calculating

$$(x^7, \cos x) = \int_{-\pi}^{\pi} \underbrace{\underset{\text{odd}}{x^7} \; \underset{\text{even}}{\cos x}}_{\text{odd}} \, dx = 0$$

2) $f(x) x^7 = c_1 \cdot 1 \cdot x^7 + c_2 x^7 \cdot x^7 + c_3 \cos x \cdot x^7$

$$\int_{-\pi}^{\pi} f(x) x^7 \, dx = 0 + c_2 \int_{-\pi}^{\pi} x^{14} \, dx + 0$$

Solve for c_2.

Prob 18, 19: Hint: $\sum$ Sines odd

$\sum$ Cosines even

Crib sheet: Formula for sol^n to Dirichlet problem on disc of radius R.

$$\Delta u = 0 \text{ on disc.}$$

$$u(R,\theta) = f(\theta)$$

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty}\left(a_n\left(\frac{r}{R}\right)^n \cos n\theta + b_n\left(\frac{r}{R}\right)^n \sin n\theta\right)$$

Fourier coeff for $f(\theta)$

Warning: $3 + \cos 7\theta + 12\sin 13\theta$

is its own Fourier Series.