1. (15) (i) Find all values of z such that $e^{iz} = 1 + i\sqrt{3}$. Write your answer in $a + ib$ form.

$e^{i(x+iy)} = e^{-y}(\cos x + i \sin x) = 1 + i\sqrt{3}$

$e^{-y} = \sqrt{1+3} = 2, \quad y = -\operatorname{Ln} 2.$

$\cos x = \frac{1}{2}, \quad \sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3} + 2k\pi$

Answer :

$z = \frac{\pi}{3} + 2k\pi - i\operatorname{Ln}2, \; k = 0, \pm 1, \pm 2, \ldots$

(15) (ii) Find all values of z such that $z^3 = -8i$. Write your answer in $a + ib$ form.

$|z^3| = 8 \Rightarrow |z| = 2$

$\arg(z^3) = \frac{3\pi}{2} + 2k\pi \Rightarrow \arg(z) = \frac{\pi}{2} + \frac{2k\pi}{3}$

$k = 1, 2, 3$

$k = 0 \Rightarrow z = 2i$

$k = 1 \Rightarrow z = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\sqrt{3} - i$

$k = 2 \Rightarrow z = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{3} - i$

Answer :

$z = 2\left(\cos\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right)\right), \; k = 0, 1, 2$

<u>OR</u>: $z = 2i, \; \pm\sqrt{3} - i$

2. (15) (i) Evaluate $\int_C \frac{e^{\sin z} + e^{\bar{z}}}{z^2}\, dz$ where C is the circle $|z| = 1$ traversed once counterclockwise.

$$I_1 = \int_C \frac{e^{\sin z}}{z^2}\, dz = 2\pi i \left(e^{\sin z}\right)'\Big|_{z=0} =$$
$$2\pi i \cos z\, e^{\sin z}\Big|_{z=0} = 2\pi i$$
$$I_2 = \int_C \frac{e^{\bar{z}}}{z^2}\, dz = \int_0^{2\pi} \frac{e^{e^{-it}}}{e^{2it}}\, d(e^{it}) = \int_0^{2\pi} e^{e^{-it}} i e^{-it}\, dt$$
$$= -e^{e^{-it}}\Big|_0^{2\pi} = 0.$$ Alternatively, $\bar{z} = \frac{1}{z}$ on C.
$$I_2 = \int_{|z|=R} \frac{e^{1/z}}{z^2}\, dz \to 0 \text{ as } R \to \infty.$$

Answer : $2\pi i$

(10) (ii) Let L be the line segment from $1 + i$ to $3 + 3i$. Evaluate $\int_L |z|^2\, dz$. Write your answer in $a + ib$ form.

$$z = (1+i)t, \quad 1 \le t \le 3$$
$$\int_1^3 t^2 |1+i|^2 (1+i)\, dt = 2(1+i)\frac{t^3}{3}\Big|_1^3 = \frac{2(1+i)(27-1)}{3}$$
$$= \frac{52}{3}(1+i)$$

Answer : $\frac{52}{3} + \frac{52}{3} i$

3. (15) (i) Find the radius of convergence R of the power series $\sum_{n=0}^{\infty} \frac{1}{(1+3i)^n} z^{2n}$.

$$\sum_{n=0}^{\infty} \left(\frac{z^2}{1+3i}\right)^n \text{ converges when}$$

$$\left|\frac{z^2}{1+3i}\right| < 1 \Rightarrow |z^2| < |1+3i| = \sqrt{10}$$

$$\Rightarrow |z| < \sqrt[4]{10}$$

Alternatively, ratio test may be applied.

Answer : $R = \sqrt[4]{10}$

(15) (ii) Find the analytic function to which the power series in (i) converges for $|z| < R$.

$$f(z) = \frac{1}{1 - \frac{z^2}{1+3i}} = \frac{1+3i}{1+3i-z^2}$$

(as the sum of a geometric series).

Answer : $f(z) = \frac{1+3i}{1+3i - z^2}$

4. (15) For which values of $R > 0$ the integral $\int_C \frac{dz}{(z^2 - 5z + 6)}$, where C is the circle $|z| = R$ traversed once counterclockwise, is equal to zero?

$$\frac{1}{z^2-5z+6} = \frac{1}{(z-2)(z-3)}$$

is not analytic at $z=2$ and $z=3$.

For $0<R<2$ the integral is equal to zero by Cauchy Theorem.

For $R>3$ the integral is equal to zero because $\left|\frac{1}{z^2-5z+6}\right| \sim \frac{1}{R^2}$ as $R \to \infty$.

For $2<R<3$,

$$\int_C \frac{dz}{(z-2)(z-3)} = \left.\frac{2\pi i}{z-3}\right|_{z=2} = -2\pi i \neq 0$$

Answer :

$0<R<2$ and $R>3$