Lesson 10 on 10.8 Applications   HWK 3: 6,7,8 due tonight
9,10,11 due Wed. next
Do before Exam

Exam 1 covers Lessons 1-11.
Tues 8-9 pm in EE-129. (1 Crib Sheet both sides.)

$$z = f(x,y), \quad \vec{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + f(x,y)\,\hat{k}$$

$$\vec{N} = \vec{r}_x \times \vec{r}_y = -\frac{\partial f}{\partial x}\hat{\imath} - \frac{\partial f}{\partial y}\hat{\jmath} + \hat{k}$$

upward

$$dA = \|\vec{r}_x \times \vec{r}_y\|\,dx\,dy$$

$$\iint_S \vec{F}\cdot\hat{n}\,dA = \iint_R \vec{F}(\vec{r}(x,y))\cdot\left[-f_x\hat{\imath} - f_y\hat{\jmath} + \hat{k}\right]dx\,dy$$

$$\hat{n} = \frac{\vec{N}}{\|\vec{r}_x \times \vec{r}_y\|} \qquad dA = \|\vec{r}_x \times \vec{r}_y\|\,dx\,dy$$

__Heat Eqn:__ $\dfrac{\partial u}{\partial t} = c\,\Delta u$

Steady State: $\dfrac{\partial u}{\partial t} \to 0.$

$$\Delta u \equiv 0 \leftarrow u \text{ harmonic}$$

Laplace Eqn.

__Famous Problem:__ Dirichlet Problem

Given a fcn φ on S, find a harmonic fcn u on Ω that has boundary values φ.

$$\begin{cases} \Delta u \equiv 0 & \text{on } \Omega \\ u\big|_S = \varphi \end{cases}$$

Given temp on surface, find steady state temp inside.

Green's Identities:

$$\operatorname{Div}(f\nabla g) = f\underbrace{(\operatorname{Div}\nabla g)}_{\Delta g} + \nabla f\cdot\nabla g$$

$$-\operatorname{Div}(g\nabla f) = g\,\Delta f + \nabla g\cdot\nabla f$$

$$\operatorname{Div}(f\nabla g - g\nabla f) = f\Delta g - g\Delta f$$

Div Thm: $$\iiint_\Omega f\Delta g - g\Delta f\, dV = \iint_S (f\nabla g - g\nabla f)\cdot\hat{n}\, dA$$

Remember: $\nabla g\cdot\hat{n} = D_{\hat{n}}\, g = \frac{\partial g}{\partial n}$ ← normal derivative

Green's Identity: $$\iiint_\Omega f\Delta g - g\Delta f\, dV = \iint_S f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\, dA$$

Consequences: Suppose g harmonic ($\Delta g \equiv 0$) and $f = 1$.

$$0 = \iiint_\Omega \underbrace{\Delta g}_{0}\, dV = \iint_S \frac{\partial g}{\partial n}\, dA$$ ← Flux heat flow $= 0$.

Back to beginning: $\operatorname{Div}(g\nabla g) = g\,\Delta g + \nabla g\cdot\nabla g$

Div. Thm: $$\iiint_\Omega g\Delta g + \nabla g\cdot\nabla g\, dV = \iint_S g\nabla g\cdot\hat{n}\, dA$$

Suppose g harmonic: (*) $$\iiint_\Omega \|\nabla g\|^2\, dV = \iint_S g\frac{\partial g}{\partial n}\, dA$$

↑ Dirichlet integral

Consequence: Sol^n to D. Prob is unique (if it exists).

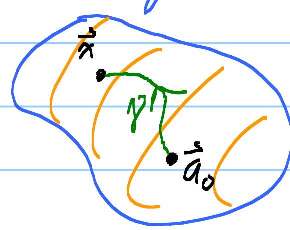
Why: Suppose u_1 and u_2 solve D. Prob. for bndry value φ. Aha! Then $u_1 - u_2$ is a harmonic fcn on Ω that is <u>zero</u> on S'! Let $g = u_1 - u_2$

(*) shows that $\iiint_\Omega \|\nabla g\|^2 \, dV = 0$.

Conclude that $\|\nabla g\|^2 \equiv 0$ on Ω. So $\nabla g \equiv 0$. g must be constant. Since $g = 0$ on S', that constant must be zero. So $g \equiv 0$. $u_1 \equiv u_2$.

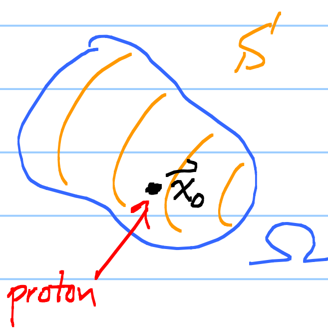
<u>Why</u> $\nabla g \equiv 0 \Rightarrow g = \text{const.}$

$\vec{a}_0$ fixed. $\vec{x}$ moving



$$0 = \int_\gamma \nabla g \cdot d\vec{r} = g(\vec{x}) - g(\vec{a}_0)$$

<u>Gauß' Law in Electrostatics:</u>



Coulomb's Law

$$\vec{E} = c \, \frac{1}{\|\vec{r}\|^2} \cdot \frac{\vec{r}}{\|\vec{r}\|}$$

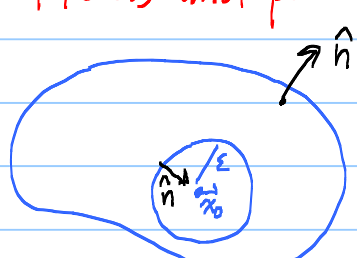
$$= K \, \nabla \left(\frac{1}{\|\vec{r}\|} \right)$$

electric potential $\leftarrow$ harmonic

$$\text{Flux}_{S'} \vec{E} = \iint_{S'} \vec{E} \cdot \hat{n} \, dA \overset{?}{=} \iiint_\Omega \underbrace{\text{Div} \, \nabla \varphi}_{0} \, dV$$

No, no, no! Fields and potential <u>blow up</u> at $\vec{x}_0$.

<u>But</u>,



$\Omega_\varepsilon = \Omega - B_\varepsilon(\vec{x}_0)$

$S'^\varepsilon = S' \cup S'_\varepsilon(\vec{x}_0)$

$$\iint_{S'^{\varepsilon}} \vec{E}\cdot\hat{n}\, dA = \iint_{S'} - \iint_{S'_{\varepsilon}(\vec{x}_0)} = \iiint_{\Omega_\varepsilon} \underbrace{\mathrm{Div}\,\nabla\varphi}_{\Delta\varphi\equiv 0}\, dV$$

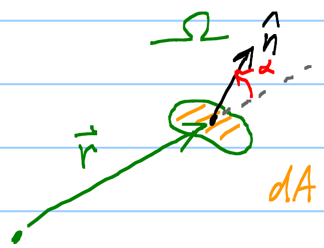
Aha! $\mathrm{Flux}_{S'}\vec{E} = \mathrm{Flux}_{S'_{\varepsilon}(\vec{x}_0)}\vec{E} = c$

Gauß' Law: $\mathrm{Flux}_{S'}\vec{E} = c\,(\text{Net charge inside } S')$

Fun application: $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k} = \vec{r}$

$$\mathrm{Div}\,\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$\iint_{S'} \vec{F}\cdot\hat{n}\, dA = \iiint_{\Omega} 3\, dV = 3\,\mathrm{Vol}(\Omega)$$



$\vec{r}\cdot\hat{n} = \underbrace{\|\vec{r}\|}\cdot 1\, \cos\alpha$

ρ in spherical coord.

$$\mathrm{Vol}(\Omega) = \frac{1}{3}\iint_{S'} \rho\cos\alpha\, dA$$

Feeling for $\mathrm{Div}\,\vec{F}$: Do Div Thm on $B_\varepsilon(\vec{x}_0)$

$$\frac{1}{\mathrm{Vol}\,B_\varepsilon}\iint_{S'_\varepsilon} \vec{F}\cdot\hat{n}\, dA = \frac{1}{\mathrm{Vol}\,B_\varepsilon}\iiint_{B_\varepsilon(\vec{x}_0)} \mathrm{Div}\,\vec{F}\, dV \xrightarrow{\varepsilon\to 0} \mathrm{Div}\,\vec{F}\Big|_{\vec{x}_0}$$

$$\mathrm{Div}\,\vec{F}\Big|_{\vec{x}_0} = \lim_{\varepsilon\to 0}\frac{1}{\mathrm{Vol}(B_\varepsilon)}\,\mathrm{Flux}_{S'_\varepsilon}\vec{F}$$

$\mathrm{Div}\,\vec{F} > 0$ fluid is being created at $\vec{x}_0$