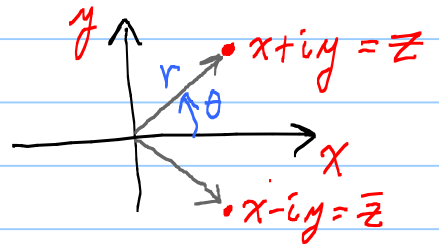
Lesson 14 on 13.1, 13.2 Complex numbers (No lecture this Friday).

Lesson 14 HWK problems <u>not</u> to be turned in. Do but not due.

HWK 4 : Lessons 9, 10, 11 due tonight 11:59 pm.

$\mathbb{C}$



$i^2 = -1$

$x = r\cos\theta$

$y = r\sin\theta$

$x + iy = r\cos\theta + i r\sin\theta = r\left(\underbrace{\cos\theta + i\sin\theta}_{e^{i\theta}}\right)$, definition

$x = \mathrm{Re}\, z \leftarrow$ Real part

$y = \mathrm{Im}\, z \leftarrow$ Imag. part

$|z| = r \leftarrow$ Modulus of z

$\bar{z} = x - iy \leftarrow$ Conjugate of z

$\theta \in \arg z \leftarrow$ θ is <u>an</u> argument of z

$\mathrm{Arg}\, z = \theta$ such that $z = |z| e^{i\theta}$ where

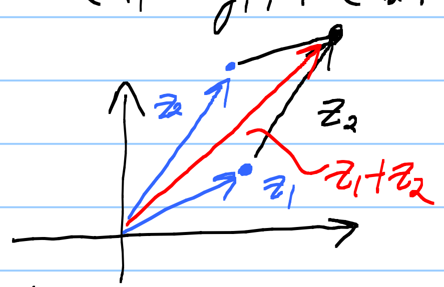
$-\pi < \theta \leq \pi$

Principal argument of z

$\arg z = \{ \mathrm{Arg}\, z + n 2\pi \; : \; n = 0, \pm 1, \pm 2, \dots \}$

<u>Complex arithmetic</u>

$+ \qquad (x_1 + i y_1) + (x_2 + i y_2) = (x_1 + x_2) + i(y_1 + y_2)$



Vector addition in $\mathbb{R}^2$

Additive inverse : $-(x+iy) = (-x) + i(-y)$.

Zero vector : $0 = 0 + i0$

$$* \quad (x_1+iy_1)(x_2+iy_2)=(x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2)$$

One vector: $\underline{1} = 1 + i0$

Multiplicative inverse: Given $a+ib \neq 0$.

Find $x+iy$ so that $(a+ib)(x+iy)=1$

$$\underbrace{(ax-by)}_{1}+i\,\underbrace{(ay+bx)}_{0} = 1+i\,0$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\det = a^2+b^2 \neq 0$

Cramer's Rule:
$$x = \frac{\det\begin{bmatrix} 1 & -b \\ 0 & a \end{bmatrix}}{\det\begin{bmatrix} a & -b \\ b & a \end{bmatrix}} = \frac{a}{a^2+b^2}$$

$$y = \frac{\det\begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}}{\det\begin{bmatrix} a & -b \\ b & a \end{bmatrix}} = \frac{-b}{a^2+b^2}$$

High school way: $x+iy = \frac{1}{a+ib}\,\frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}$ ✓

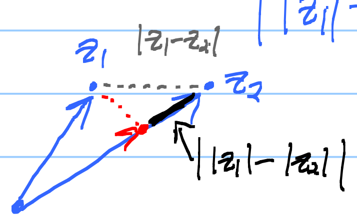
$\mathbb{C}$ is a field.

Laws: $\overline{(z_1 z_2)} = \bar{z}_1\,\bar{z}_2$

$$|z_1 z_2| = |z_1|\,|z_2|$$

Triangle Ineqs: $|z_1+z_2| \leq |z_1| + |z_2|$

$$\big|\,|z_1| - |z_2|\,\big| \leq |z_1 - z_2|$$



$z = x + iy$: $|x|$ or $|y| \le |z| \le |x| + |y|$

Important Fact: $e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}$

Why: $e^{i\alpha} e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$$= \Big[\underbrace{\cos\alpha\cos\beta \overset{i^2=-1}{-} \sin\alpha\sin\beta}_{\cos(\alpha+\beta)}\Big] + i\Big[\underbrace{\cos\alpha\sin\beta + \sin\alpha\cos\beta}_{\sin(\alpha+\beta)}\Big]$$

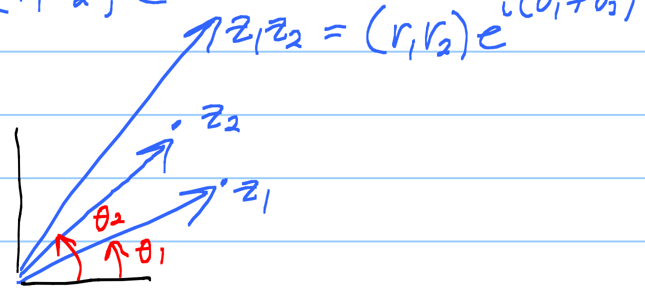
de Moivre's formula: $(re^{i\theta})^N = r^N(\cos N\theta + i\sin N\theta)$

Note: Only uses fact plus $z_1 z_2 = z_2 z_1$ (Comutivity)

Geometry of complex mult. $z_1 = r_1 e^{i\theta_1}$

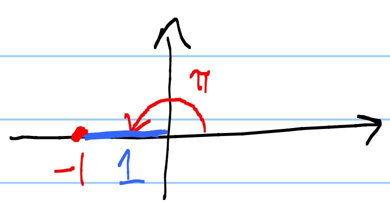
$z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = (r_1 r_2) e^{i(\theta_1+\theta_2)}$



Prob: Find all complex #'s z such that $z^3 = -1$

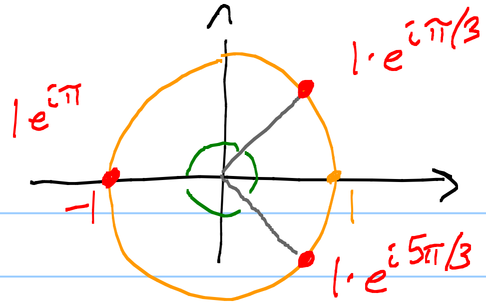
Let $z = re^{i\theta}$, $z^3 = \underset{r>0}{r^3} e^{i3\theta} \overset{\text{want}}{=} \underbrace{1 \cdot e^{i\pi}}_{=-1}$



$r^3 = 1$ $(r>0)$ $\boxed{r=1}$

θ problem: Need $3\theta = \pi + 2n\pi$, $n = 0, \pm1, \pm2, \ldots$

$$\boxed{\theta = \frac{\pi}{3} + n\frac{2\pi}{3}} \quad n = 0, \pm1, \pm2, \ldots$$



Only 3 sol'ns

-1

$\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Algebra: $Az^2 + Bz + C = 0$

$z^2 + 1 = 0$

$z^2 = -1$

Toss in $i = \sqrt{-1}$.

Step 1: Factor out A:

$\rightarrow z^2 + bz + c = 0$

where $b = \frac{B}{A}$, $c = \frac{C}{A}$.

Step 2: Complete square

$$\left(z + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) = 0$$

$$\left(z + \frac{b}{2}\right)^2 = \left(\frac{b^2}{4} - c\right)$$

$$z + \frac{b}{2} = \pm\sqrt{\frac{b^2}{4} - c}$$

$$z = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$$

Fundamental Theorem of Algebra: $N \geq 1$, $a_N \neq 0$.

$$P(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$$

Then $P(z)$ has a root in $\mathbb{C}$.

Consequence: Poly's factor over $\mathbb{C}$.

Calculus with complex #'s.