Lesson 15 on 13.3 Analytic functions

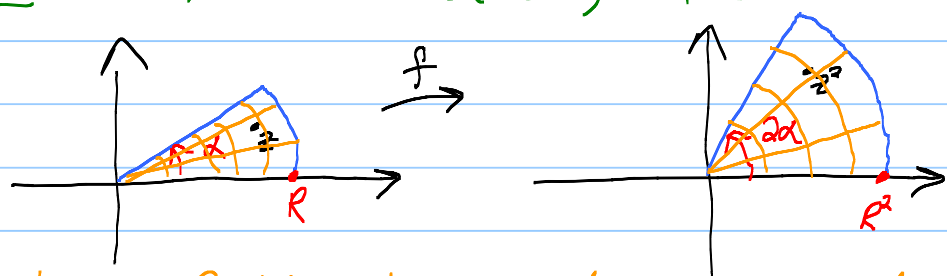
No WebEx this week.

Lesson 14: Do but not due. HWK 5: Lessons 15, 16, 17 due next Wed.

Office Hours: M, T, W 2-3 pm in MATH 750

Complex functions $\mathbb{C} \to \mathbb{C}$

EX: $f(z) = z^2$ $\quad f(re^{i\theta}) = r^2 e^{i2\theta}$



Hmmm. Right angles in grid are preserved.

Conformal?

$$f(x+iy) = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$$

Hmmm. x^2-y^2 and $2xy$ are harmonic ($\Delta u \equiv 0$)

$\mathbb{C} \to \mathbb{C}$: $f(z) = z^2$

$\mathbb{R}^2 \to \mathbb{R}^2$: $(x,y) \longmapsto (u(x,y), v(x,y))$

Same picture in plane.

Limits: $\lim_{z \to a} f(z) = L \in \mathbb{C}$ means:

Given $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(z) - L| < \varepsilon$ when $|z-a| < \delta$, $z \neq a$.

Note: f does not need to be defined at a for limit to make sense.

EX: $\dfrac{z^2-a^2}{z-a} \leftarrow \dfrac{0}{0}$ at a. Not defined there.

but $\dfrac{z^2-a^2}{z-a} = \dfrac{(z-a)(z+a)}{z-a} = z+a \to 2a$ as $z \to a$.

Why: $\left|f(z)-L\right| = \left|\dfrac{z^2-a^2}{z-a} - 2a\right|$

$$= \left|(z+a) - 2a\right|$$

$$= |z-a| < \varepsilon$$

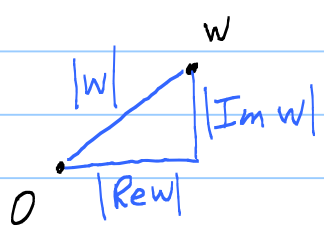
want

Aha! Need $|z-a| < \delta$ where $\delta = \varepsilon$.

Fact: $f(x+iy) = u(x,y) + i\,v(x,y)$ $\quad z_0 = x_0 + i y_0$

$$\lim_{z\to z_0} f(z) = L \iff \begin{cases} \displaystyle\lim_{(x,y)\to(x_0,y_0)} u(x,y) = \operatorname{Re} L \\ \displaystyle\lim_{(x,y)\to(x_0,y_0)} v(x,y) = \operatorname{Im} L \end{cases}$$

Why:



Think $w = f(z) - L$

$$\begin{cases} |\operatorname{Re} w| \le |w| \\ |\operatorname{Im} w| \le |w| \end{cases}$$

$$|w| = \sqrt{(\operatorname{Re} w)^2 + (\operatorname{Im} w)^2}$$

EX: $e^z = e^{x+iy} = e^x e^{iy}$

$$= e^x(\cos y + i \sin y)$$

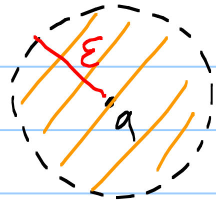
Complex exponential $\quad e^z = \underbrace{(e^x \cos y)}_{u(x,y)} + i\,\underbrace{(e^x \sin y)}_{v(x,y)}$

Def^n

Continuity: f is continuous at a means

$$\lim_{z\to a} f(z) = f(a).$$

Notation: $D_\varepsilon(a) = \{z : |z-a| < \varepsilon\}$

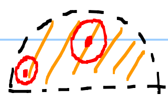


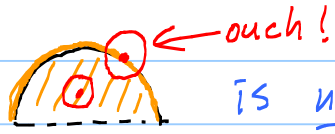
Open disc of radius ε about a.

Def^{n}: Ω is an open set in $\mathbb{C}$ if:

For any point z_0 in Ω, there is an $\varepsilon > 0$ (that might depend on z_0) such that $D_\varepsilon(z_0) \subset \Omega$.

EX: $D_1(0)$ is open.

EX:  is open.

EX: 

is not open

Def^{n}: z_0 is called a boundary point of S

if: Given $\varepsilon > 0$, no matter how small, $D_\varepsilon(z_0)$ contains a point in S and a point in $\mathbb{C} - S$.

Def^{n}: S is closed if it contains all its boundary points.

$\overline{D_\varepsilon(a)} = \{z : |z-a| \le \varepsilon\}$

closed.

Remark: open $\neq$ (not closed)

EX:  $\leftarrow$ not closed, not open.

Fact: S is closed $\Longleftrightarrow$ $\mathbb{C} - S$ is open.

Def^n: $f(z)$ on an open set Ω. f is complex diff'ble at $a \in \Omega$ means

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \text{ exists at } a.$$

Write $f'(a) =$ the limit. ← Complex derivative.

Remark: Same as saying $\displaystyle\lim_{\Delta z \to 0} \frac{f(a + \Delta z) - f(a)}{\Delta z} = f'(a)$.

EX: $f(z) = z^2$. $\quad DQ = \dfrac{z^2 - a^2}{z - a} = z + a \to 2a$ as $z \to a$.

So $f'(a) = 2a$.

Fact: All the basic rules of limits and derivatives you know $\mathbb{R} \to \mathbb{R}$ carry over $\mathbb{C} \to \mathbb{C}$. Proofs exactly the same:

$$\left.\begin{array}{l} \mathbb{R}: |x| = \text{absolute value} \\ \mathbb{C}: |z| = \text{modulus of } z. \end{array}\right\} \Delta \text{ ineq.}$$

e.g. P, Q polys. $\displaystyle\lim_{z \to a} \frac{P(z)}{Q(z)} = \frac{P(a)}{Q(a)}$ provided $Q(a) \neq 0$.

e.g. $f(z) = 7z^5 + 3z^4 + z + 1$

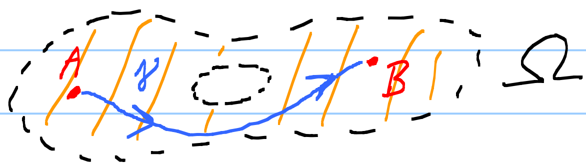
$f'(z) = 7 \cdot 5 z^4 + 3 \cdot 4 z^3 + 1 + 0$

Biggies: $\quad (fg)' = f'g + fg'$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$ provided that $g(z) \neq 0$.

Def^n: Ω connected open set means:

Ω is open and any two points in Ω can be connected by a continuous curve γ that stays completely in Ω.



Def^n: Connected open sets are called domains.

Def^n: f is analytic on a domain Ω if it is complex diff'ble at each point in Ω.

Ques: Is e^z analytic on $\mathbb{C}$?

EX: z^2 is. But $f(z) = \mathrm{Re}\, z$ is not.