Lesson 16 on 13.4 Cauchy-Riemann Equations

No office hour today. Thurs. 2-3 pm instead

Lesson 14 problems: Do, but not to be turned in.

HWK 5: Lessons 15, 16, 17 due Wed, next.

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

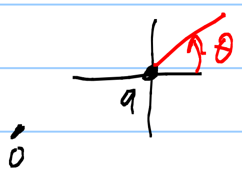
EX: $f(z) = \bar{z} = x - iy$ is not complex diff'ble

$$DQ = \frac{\bar{z} - \bar{a}}{z - a} = \frac{\varepsilon e^{-i\theta}}{(\varepsilon e^{i\theta})}$$

$\longleftarrow$ conjugate of $\varepsilon e^{i\theta}$

$$= \frac{e^{-i\theta}}{e^{i\theta}} = e^{-i\theta - (i\theta)} = e^{-2i\theta}$$

Ouch! Different limit values for diff. directions.



Cauchy-Riemann Eqns: If $f(x+iy) = u(x,y) + iv(x,y)$ is complex diff'ble at $z_0 = x_0 + iy_0$, then

1) the first partials of u, v exist at (x_0, y_0), and

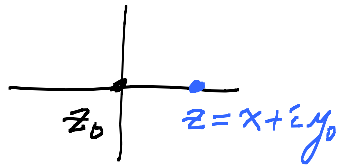
2)
$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \text{C-R Eqns}$$

EX: $\bar{z} = x - iy$ $\quad u(x,y) = x$, $\quad v(x,y) = -y$

$$1 = u_x \overset{?}{=} v_y = -1 \quad \underline{\text{no!}}$$

(other C-R Eqn holds).

Why:



$$DQ = \frac{f(x+iy_0) - f(x_0+iy_0)}{\underbrace{(x+iy_0) - (x_0+iy_0)}_{x-x_0}}$$

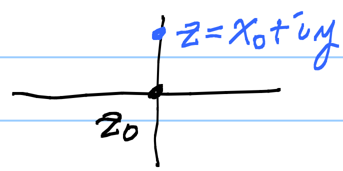
$$= \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i\,\frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}$$

$DQ \to f'(z_0) \iff$ Re and Im parts have limits.

See first partial u_x, v_x exist and

$$\boxed{f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0).}$$

Next



$$DQ = \frac{u(x_0,y) - u(x_0,y_0)}{i(y-y_0)} + i\,\frac{v(x_0,y) - v(x_0,y_0)}{i(y-y_0)}$$

$$\longrightarrow \frac{\partial v}{\partial y}(x_0,y_0) - i\,\frac{\partial u}{\partial y}(x_0,y_0)$$

$$\boxed{f' = v_y - i\,u_y}$$

Equate red boxes to get C-R Eqns.

Fact: If complex derivative exists, then

$$f' = \begin{cases} u_x + i\,v_x \\ v_y - i\,u_y \end{cases}$$

EX: $e^z = \underbrace{e^x \cos y}_{u} + i\, \underbrace{e^x \sin y}_{v}$

$u_x = e^x \cos y \overset{?}{=} v_y = e^x \cos y$ ✓

$u_y = -e^x \sin y \overset{?}{=} -v_x = -(e^x \sin y)$ ✓

Is e^z analytic? Yes, because

Theorem: If u, v are C^1-smooth and satisfy C-R Eqns, then $f = u + iv$ is analytic. [Ω a domain on which u, v C^1.]

Why: Taylor's Thm:

$$u(x,y) = \underbrace{u(x_0,y_0)}_{u^0} + \underbrace{u_x(x_0,y_0)}_{u_x^0}(x-x_0) + \underbrace{u_y(x_0,y_0)}_{u_y^0}(y-y_0) + R_u(x,y)$$

Similarly for v.

Taylor: $$\lim_{(x,y)\to(x_0,y_0)} \frac{R_u(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Note: $\sqrt{\quad} = |z - z_0|$

$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{(u+iv)-(u^0+iv^0)}{z-z_0}$$

$$= \frac{[u_x^0(x-x_0) + \underbrace{u_y^0}_{=-v_x^0}(y-y_0)] + i[v_x^0(x-x_0) + \underbrace{v_y^0}_{=u_x^0}(y-y_0)]}{z-z_0}$$

$$+ \frac{R_u + iR_v}{z-z_0}$$

Aha! Complex multiplication on top!

$$DQ = \frac{[u_x^0 + i v_x^0]\,[\cancel{(x-x_0)+i(y-y_0)}]}{\cancel{z-z_0}} + R$$

$$= u_x^0 + i v_x^0 + \frac{R_u + i R_v}{z-z_0}$$

$$\Big| DQ - \underbrace{(u_x^0 + i v_x^0)}_{= f'(z_0)} \Big| = \left| \frac{R_u + i R_v}{z-z_0} \right| \le \frac{|R_u|}{|z-z_0|} + \frac{|R_v|}{|z-z_0|} \to 0$$

as $z \to z_0$!

So e^z is analytic on $\mathbb{C}$ $\leftarrow$ Entire.

Consequences of CR Eqns.

1) If $f'(z) \equiv 0$ on a domain, then $f(z)$ is a constant fcn.

Why: $f' = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} = 0 \implies \begin{matrix} \nabla u \equiv 0 \\ \nabla v \equiv 0 \end{matrix}$

$\implies$ u, v const.

$$0 = \int_\gamma \nabla u \, d\vec{r} = u(z) - u(z_0) \leftarrow z_0 \text{ fixed.}$$

z moves around

So $u(z) \equiv u(z_0)$.

2) If $|f(z)| \equiv C$, f analytic.

Then $f(z)$ is a constant fcn.

Why: (*) $u^2 + v^2 = K$, a const.

Case $K=0$. Then $u\equiv 0, v\equiv 0$. $f\equiv 0$.

Case $K>0$.

$$\frac{\partial}{\partial x}(*):\ \ 2u\,u_x+2v\,v_x=0$$
$$\frac{\partial}{\partial y}(*):\ \ 2u\,\underbrace{u_y}_{=-v_x}+2v\,\underbrace{v_y}_{=u_x}=0$$

$$\underbrace{\begin{bmatrix} u & v \\ v & -u \end{bmatrix}}_{\det=-u^2-v^2=-K\neq 0.}\begin{pmatrix} u_x \\ v_x\end{pmatrix}=\vec{0}$$

Cramer's Rule: $\begin{cases} u_x=0 \\ v_x=0\end{cases}$ C-R Equs $\Rightarrow$ y derivatives $=0$ too.

So $\nabla u\equiv 0$, $\nabla v\equiv 0$, and u,v are const.

3) f analytic and u,v are C^2-smooth, then u and v must be <u>harmonic</u>.

<u>Why</u>: $\begin{cases} u_x=v_y & (A) \\ u_y=-v_x & (B)\end{cases}$

$\frac{\partial}{\partial x}(A)$: $u_{xx}=v_{xy}$

$\frac{\partial}{\partial y}(B)$: $u_{yy}=-v_{yx}$

$\checkmark$ C^2 $\Rightarrow$ mixed partials equal

So $u_{xx}=-u_{yy}$ $\underbrace{\Delta u=u_{xx}+u_{yy}\equiv 0.}_{u\text{ harmonic!}}$

Similarly, $\Delta v\equiv 0$.

Problem: Given $\mathcal{C}^2$-smooth harmonic fcn u, can we find harmonic v so that $u+iv$ is analytic?

Yes, if domain given is simply connected.

EX: $u = e^x \cos y + xy$ harmonic ✓

Want v with $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$$\begin{cases} v_x \overset{\text{want}}{=} -u_y = -\left(e^x(-\sin y) + x\right) = e^x \sin y - x & (A) \\ v_y = u_x = e^x \cos y + y & (B) \end{cases}$$

C-R eqns $\Longleftrightarrow$ Curl $\vec{F} = 0$!

Use (A): $v = \int e^x \sin y - x \, dx$

$$= e^x \sin y - \tfrac{1}{2}x^2 + g(y)$$

Use (B): $\frac{\partial}{\partial y}\Big[\underbrace{e^x \sin y - \tfrac{1}{2}x^2 + g(y)}_{v}\Big] \overset{\text{want}}{=} e^x \cos y + y$

$$e^x \cos y + 0 + g'(y) = e^x \cos y + y$$

$$g'(y) = y$$

So $g(y) = \tfrac{1}{2}y^2 + C$

Done $v = e^x \sin y - \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + C$