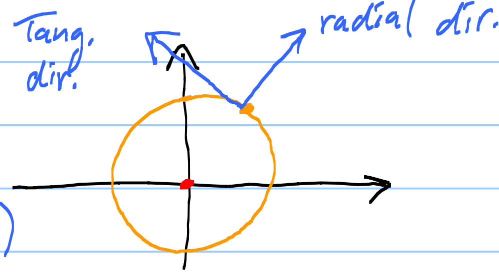
Lesson 17 on 13.5 the complex exponential function

WebEx office hour Tues 8-9 pm. Regular office hours T, W 2-3 pm in MATH 750

HWK 5: Lessons 15, 16, 17 due Wed.

Polar C-R Eqns:



$$f(re^{i\theta}) = u(r,\theta) + i v(r,\theta)$$

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{cases}$$

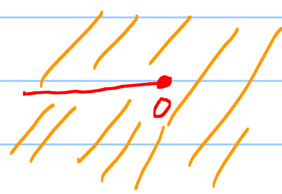
EX: $f(re^{i\theta}) = \underbrace{\ln r}_{u} + i \underbrace{\theta}_{v} \qquad -\pi < \theta < \pi$

$$u_r = \frac{1}{r} \overset{?}{=} \frac{1}{r} v_\theta = \frac{1}{r}\cdot 1 \quad \checkmark$$

$$v_r = 0 \overset{?}{=} -\frac{1}{r}\frac{\partial u}{\partial \theta} = 0 \quad \checkmark$$

$f(z) = \ln|z| + i \operatorname{Arg} z \leftarrow$ Complex log fcn.

f analytic on



$\mathbb{C} - (-\infty, 0]$

Complex Exponential fcn $E(z) = e^z$

$$E(x+iy) = e^x \cos y + i e^x \sin y$$

Properties: 1) C-R Eqns show that $E(z)$ is analytic on $\mathbb{C}$ (entire)

$$\left.\begin{aligned} &2)\ E'(z) = E(z) \\ &3)\ E(0) = 1 \end{aligned}\right\} \text{ Characterize } E(z).$$

4) $E(z_1+z_2) = E(z_1)E(z_2)$

5) $E(-z) = 1/E(z)$ ← $E(z)$ is non-vanishing!

$$E(z_1 - z_2) = \frac{E(z_1)}{E(z_2)}$$

6) $E(z) = 1 \iff z = 2n\pi i \quad n = 0, \pm 1, \pm 2, \ldots$

<u>Why</u>: $E'(z) = E(z)$. $\quad E'(z) = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases}$

$\frac{\partial}{\partial x}(e^x \cos y) + i \frac{\partial}{\partial x}(e^x \sin y)$

$= u + iv = E(z)$ ✓

3) $E(0) = e^0 \cos 0 + i e^0 \sin 0 = 1 + i0 = \underline{1}$

4) $e^{x_1 + i y_1} e^{x_2 + i y_2} = e^{x_1} e^{i y_1} e^{x_2} e^{i y_2}$

$= (e^{x_1} e^{x_2})(e^{i y_1} e^{i y_2})$

$= e^{x_1 + x_2} \cdot e^{i(y_1 + y_2)}$ ← last time via Trig.

$= e^{z_1 + z_2}$

5) $1 = E(0) = E(z + (-z)) = E(z)E(-z)$

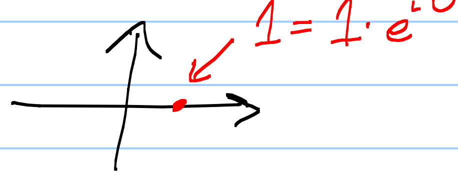
↑ can't be zero

So $E(-z) = 1/E(z)$. ✓

6) $E(z) = 1$

$\underbrace{e^x}_{\substack{\text{radius} \\ = 1}} \; e^{iy} = 1$

↑ angle $= 0, \pm 2\pi, \pm 4\pi, \ldots$



$x = 0$.

<u>Fun thing to do</u>: <u>Find</u> analytic fcn such that $E'(z) = E(z)$ and $E(0) = 1$.

$$E'(z) = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} \overset{\text{want}}{=} u + i v$$

$$\begin{cases} u_x = u : & u(x,y) = g(y)e^x \\ v_x = v : & v(x,y) = h(y)e^x \end{cases}$$

$$v_y = h'(y)e^x \overset{\text{want}}{=} u = g(y)e^x$$

(A) $\boxed{h'(y) = g(y)}$

$$u_y = g'(y)e^x \overset{\text{want}}{=} -v = -h(y)e^x$$

(B) $\boxed{g'(y) = -h(y)}$

2x2 system

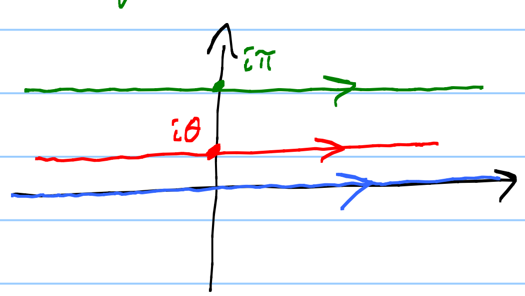
Initial Cond $E(0) = 1$. $\begin{cases} g(0) = 1 \\ h(0) = 0 \end{cases}$

$$\frac{d}{dy}(A) = h''(y) = g'(y) \overset{(B)}{=} -h(y)$$

$$\begin{cases} h''(y) + h(y) = 0 \\ h(0) = 0 \end{cases} \leftarrow h(y) = \sin y$$

(A): $g(y) = h'(y) = \cos y$

Mapping properties: e^z

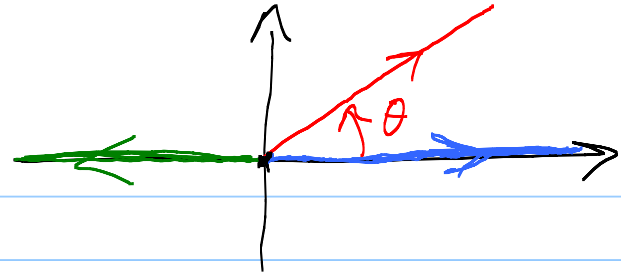


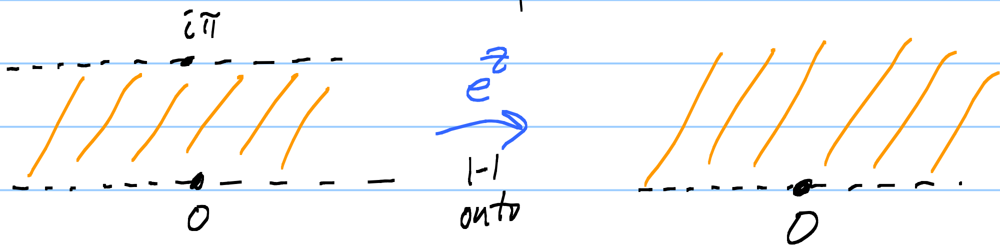
$x + i\theta$

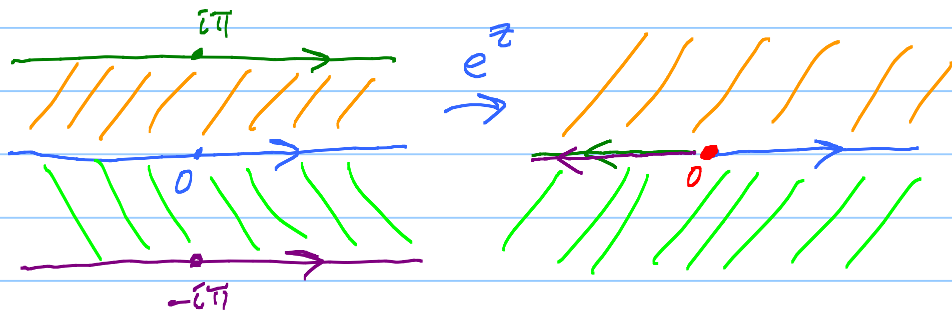
$-\infty < x < \infty$

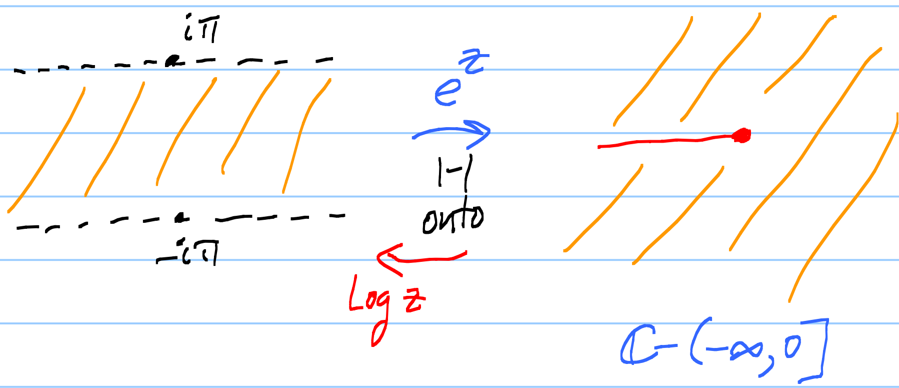
$0 < e^x < \infty$

$e^{x+i\theta} = e^x e^{i\theta}$

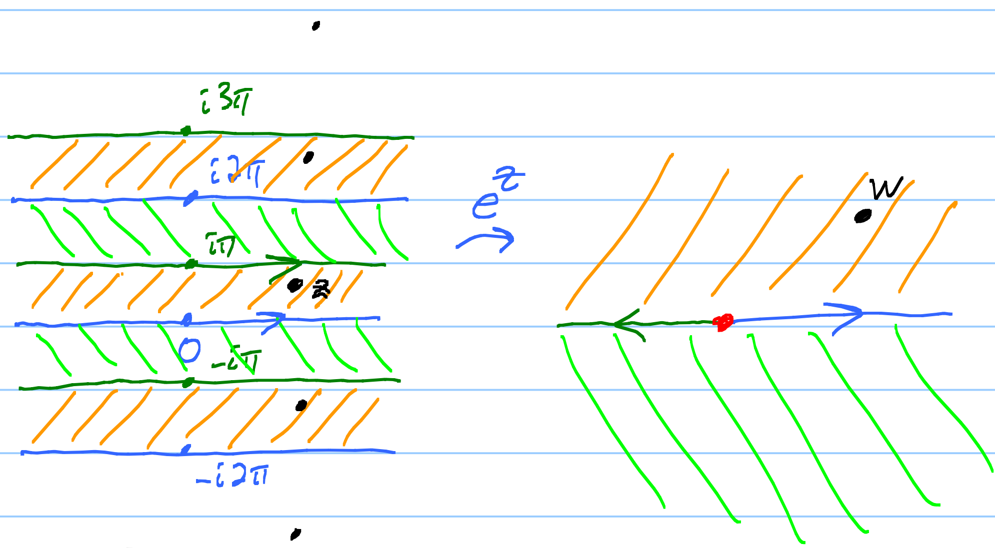








Big picture:



$z \overset{?}{=} \log w$ ← ∞ many to choose from!

$e^z = e^x e^{iy} = w$ want

See $e^x = |w|$ $\qquad y \in \arg w$

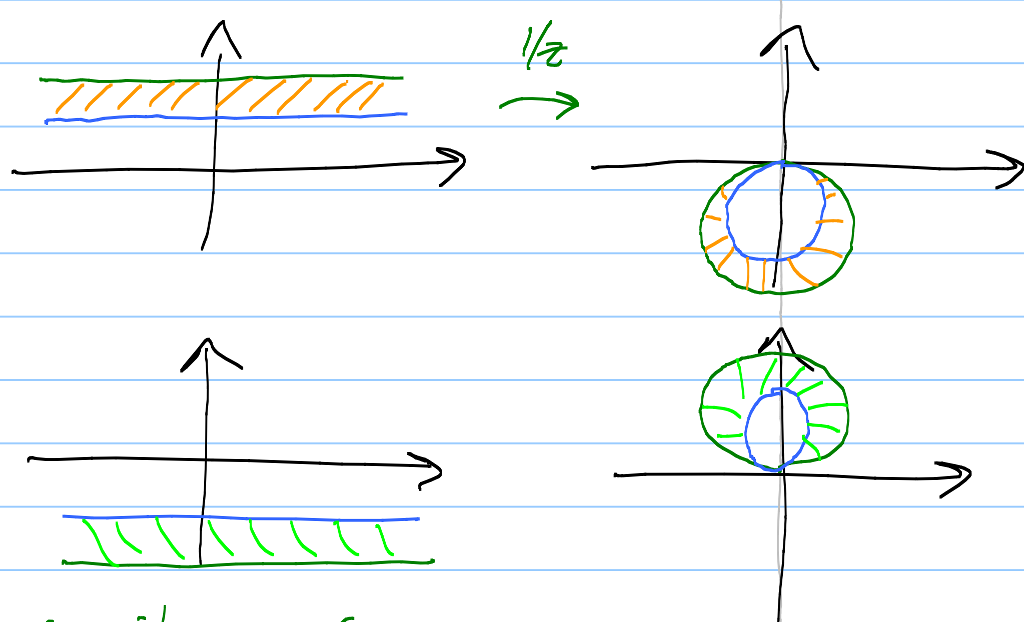
$\boxed{x = \operatorname{Ln}|w|}$ $\qquad y = \operatorname{Arg} w + n2\pi$ all work $n = 0, \pm 1, \pm 2, \ldots$

Principal branch of complex log

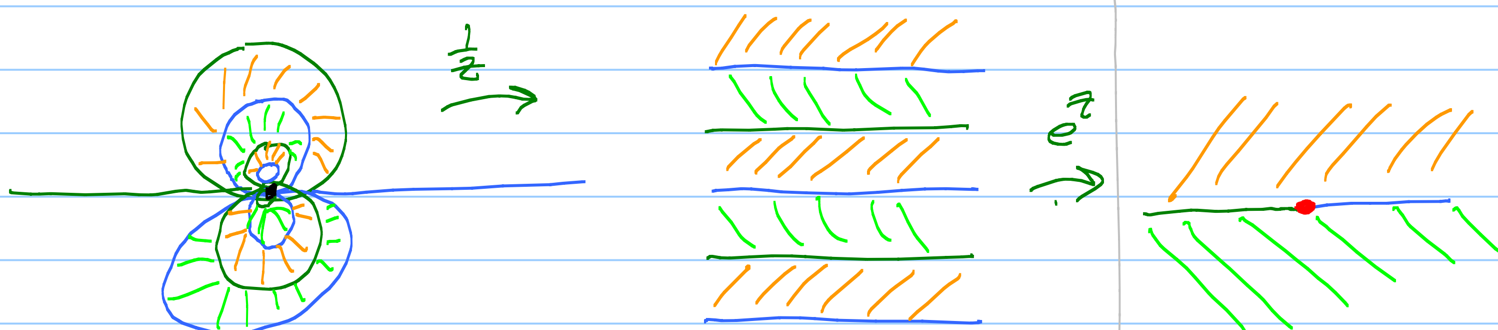
$\operatorname{Log} w = \operatorname{Ln}|w| + i \operatorname{Arg} w$ θ $\quad -\pi < \theta \le \pi$

$e^{1/z}$ has a mind blowing singularity at $z=0$

Step 1:



Fact: $\frac{1}{z}$ is its own inverse



$e^{1/z}$ shreds $\mathbb{C}$ at $z=0$. essential singularity