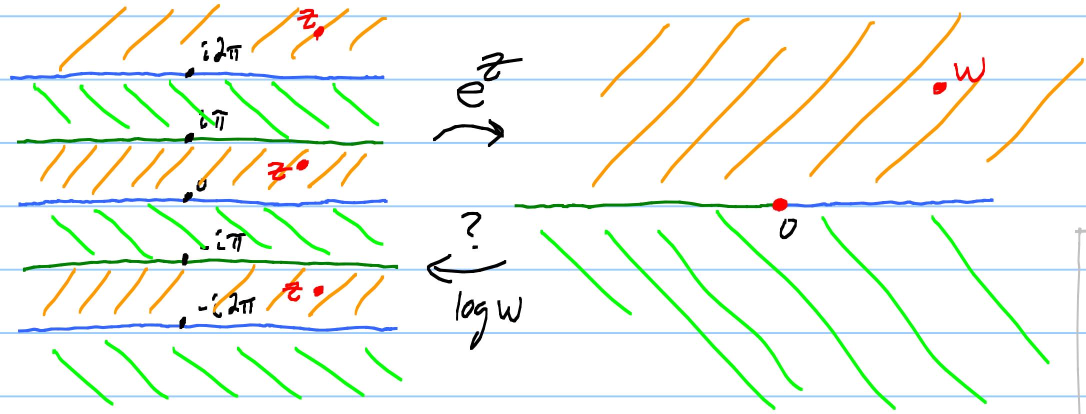
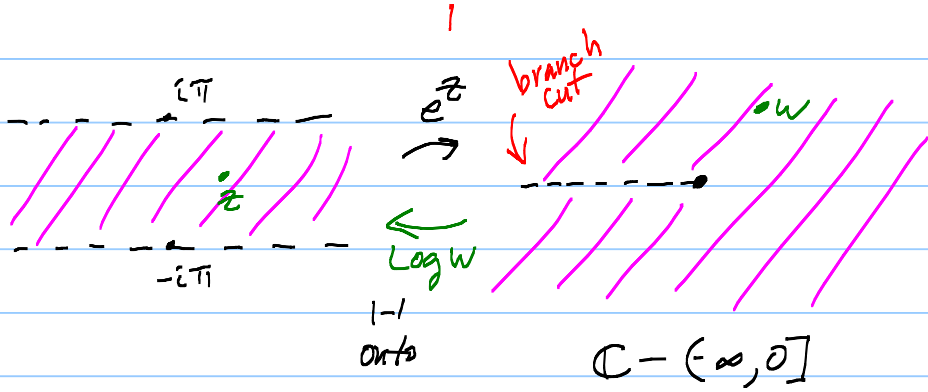
Lesson 18 on 13.6 Complex trig functions

HWK 5: Lessons 15,16,17 Wed.
WebEx Off. Hr. Tues. 8-9 pm



$$e^{z+2\pi i} = e^z \underbrace{e^{2\pi i}}_{1} = e^z$$

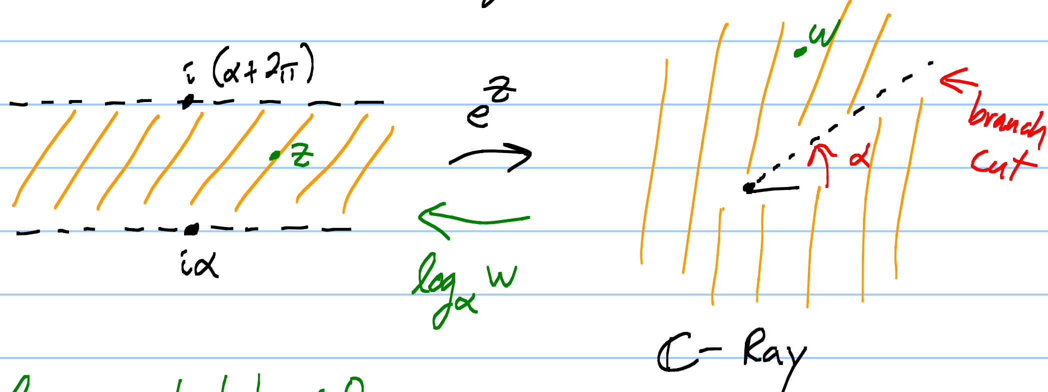


$\text{Log } w = \text{Ln}|w| + i \text{ Arg } w$ ← Principal arg.

Principal branch of complex log

$$\log w = \{z : e^z = w\} = \{\text{Ln}|w| + i\theta \,;\, \theta \in \arg w\}$$

$$= \{\text{Ln}|w| + i(\text{Arg } w + n2\pi) \,;\, n \in \mathbb{Z}\}$$



$\log_\alpha w = \text{Ln}|w| + i\theta$

where $\theta \in \arg w$ in $\alpha < \theta < \alpha + 2\pi$

Warning: $e^{z_1+z_2} = e^{z_1} e^{z_2}$ ← always true.

$\text{Log}\, w_1 w_2 = \text{Log}\, w_1 + \text{Log}\, w_2$ ← sometimes true!

$+ in2\pi$ where $n = 0$ or 1 or -1

$e^{\log w} = w$ ← always true

$\text{Log}\, e^z = z$ ← sometimes true

$+in2\pi$ for $n \in \mathbb{Z}$.

Complex Trig fcns:

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos x - i \sin x$$

$+$ $\quad 2\cos x = e^{ix} + e^{-ix}$

$-$ $\quad 2i \sin x = e^{ix} - e^{-ix}$

Def^n: $\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \quad z \in \mathbb{C}$

$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$\tan z = \frac{\sin z}{\cos z}$, etc.

Euler: $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$

Fact: Any Trig identity holds for complex angles!

EX: $\sin 2z = \frac{1}{2i}\left(e^{i(2z)} - e^{-i(2z)}\right)$

$\stackrel{?}{=} 2 \sin z \cos z$

$= 2\, \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$

Use $e^{iz} \cdot e^{iz} = e^{i2z}$, $e^{iz} \cdot e^{-iz} = e^{iz+(-iz)} = e^0 = 1$

EX: $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$$\cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$

Hmmm: $\cos iy = \frac{1}{2}\left(e^{i(iy)} + e^{-i(iy)}\right)$

$$= \frac{1}{2}\left(e^{-y} + e^{y}\right) = \cosh y$$

$$\sin iy = \frac{1}{2i}\left(e^{-y} - e^{y}\right) = -\frac{1}{i}\sinh y$$

So $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$

Complex hyperbolic trig fcns:

$$\cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right) \qquad \operatorname{Tanh} z = \frac{\sinh z}{\cosh z}$$

$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right)$$

Derivative formulas from Freshman calculus hold $\mathbb{C} \to \mathbb{C}$.

Why: $\frac{d}{dz}\left(e^{z}\right) = e^{z}$ and $\frac{d}{dz}\left(e^{iz}\right) = i e^{iz}$

and $\frac{d}{dz}\left(e^{-iz}\right) = -i e^{iz}$

[Today: See via C-R equs.]

$$\frac{d}{dz}\left(\sin z\right) = \frac{d}{dz}\left[\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right]$$

$$= \frac{1}{2i}\left(i e^{iz} - (-i) e^{-iz}\right)$$

$$= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos z \checkmark$$

Fun thing: What is complex Arc Sin fcn?

$$w = \operatorname{Arc\,Sin} z$$

$$\sin w = z$$

$$\frac{1}{2i}\left(e^{iw} - e^{-iw}\right) = z$$

$$e^{iw} - e^{-iw} = 2iz$$ ← multiply by e^{iw}

$$\left(e^{iw}\right)^2 - 1 = 2iz\left(e^{iw}\right)$$ ← quadratic eqn

$$\left(e^{iw}\right)^2 - 2iz\left(e^{iw}\right) - 1 = 0$$

Quad form: $$e^{iw} = \frac{-(-2iz) + \sqrt{(-2iz)^2 + 4}}{2 \cdot 1}$$

$$= iz + \sqrt{1 - z^2}$$

$$iw = \log\left(iz + \sqrt{1-z^2}\right)$$

$$w = -i \log\left(iz + \sqrt{1-z^2}\right)$$ ← lots of solutions

Exciting fact: Complex exponential, complex log, and algebraic fcns generate all the elementary functions.

Liouville: $\int e^{-x^2}\,dx$ is not an elem. fcn!

Prob: Find all $z \in \mathbb{C}$ such that $\cos z = 2$.

$$\frac{1}{2}\left(e^{iz}+e^{-iz}\right)=2$$

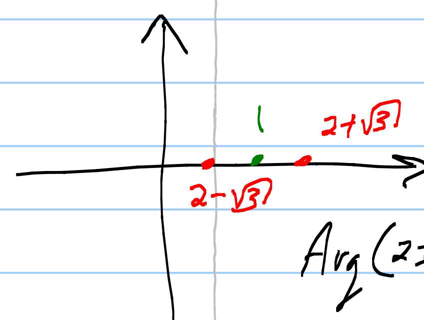
$$e^{iz}+e^{-iz}=4 \leftarrow \text{mult by } e^{iz}$$

$$\left(e^{iz}\right)^2+1=4e^{iz}$$

$$\left(e^{iz}\right)^2-4\left(e^{iz}\right)+1=0$$

$$e^{iz}=\frac{-(-4)+\sqrt{(-4)^2-4}}{2\cdot 1}$$

$$e^{iz}=2\pm\sqrt{3}$$



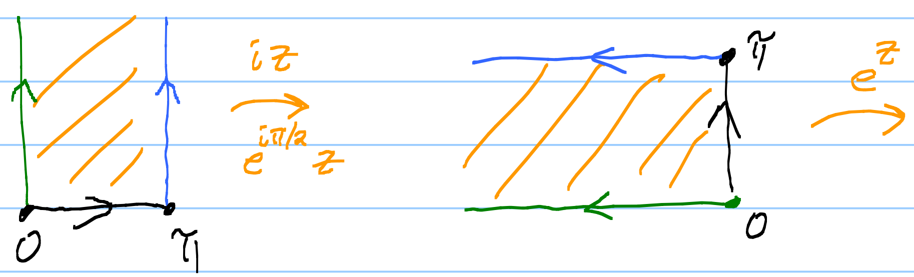
$$\operatorname{Arg}(2\pm\sqrt{3})=0$$

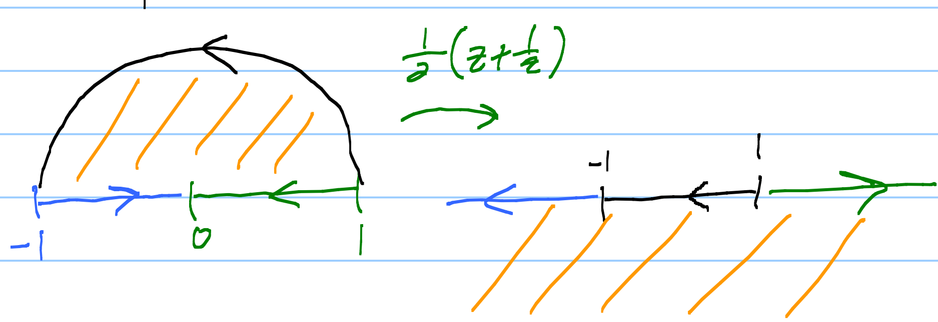
$$iz \in \log\left(2\pm\sqrt{3}\right)$$

$$z \in -i\log\left(2\pm\sqrt{3}\right)$$

$$z=-i\left(\operatorname{Ln}\left|2\pm\sqrt{3}\right|+i\left(0+2n\pi\right)\right)$$

$$=2n\pi-i\operatorname{Ln}\left(2\pm\sqrt{3}\right)$$





$$\cos(z+2\pi)=\cos z \qquad \cos(-z)=\cos z$$