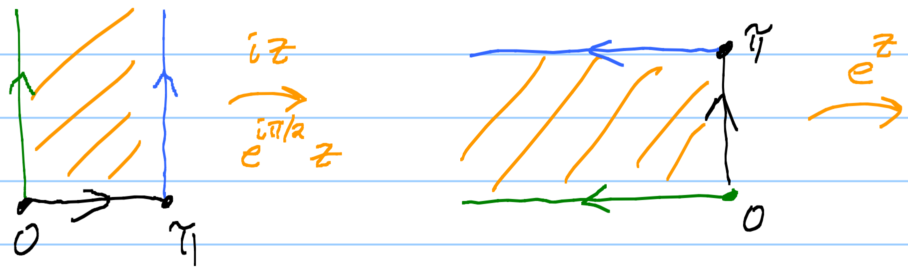
Lesson 19 on 13.7 the complex log function and powers

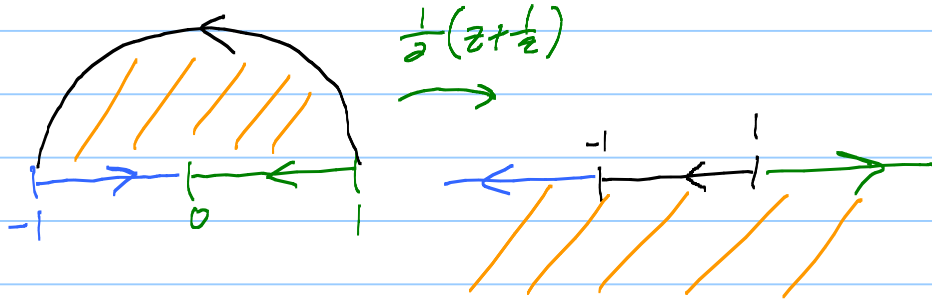
HWK 5: 15, 16, 17 due tonight

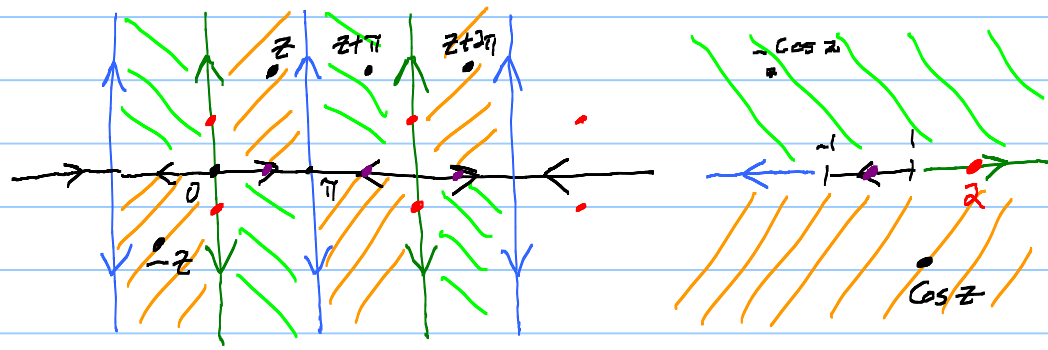
Cos(z)

$\cos(-z) = \cos z$

$\cos(z+\pi) = -\cos(z)$







$\cos z = 2, \quad z = n2\pi - i\,\mathrm{Ln}(2 \pm \sqrt{3})$

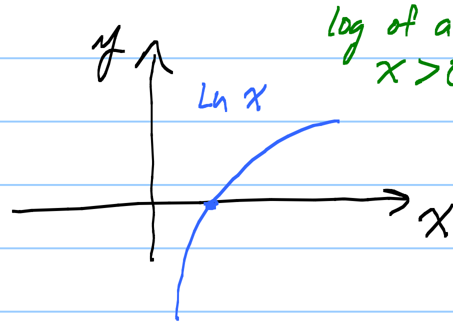
$-\mathrm{Ln}(2+\sqrt{3}) = \mathrm{Ln}(2-\sqrt{3})$

See zeroes of $\cos z$ are same as $\cos x : \mathbb{R} \to \mathbb{R}$

$\frac{\pi}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

Complex log fcn:

My notation: $\mathrm{Ln}\, x$ = real natural log of a real $x > 0$



$$e^x : (-\infty, \infty) \xrightarrow[\text{onto}]{1-1} (0, \infty)$$

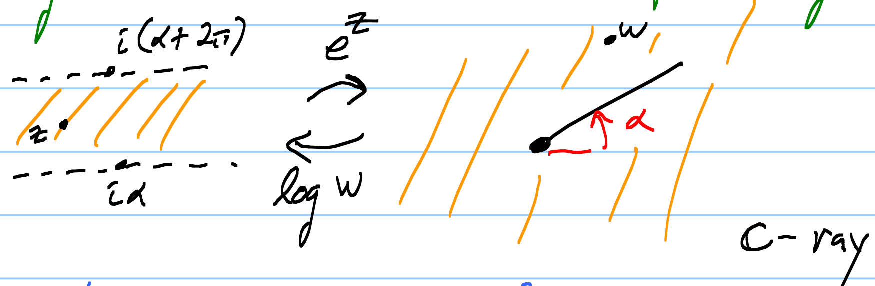
$\operatorname{Ln} x$

$\operatorname{Log} z = \operatorname{Ln}|z| + i \operatorname{Arg} z \leftarrow$ Principal log

$$\log z = \{ w : z = e^w \} = \{ \operatorname{Ln}|z| + i\theta : \theta \in \arg z \}$$

$$= \{ \operatorname{Ln}|z| + i(\operatorname{Arg} z + n2\pi) : n = 0, \pm 1, \pm 2, \dots \}$$

$\log z$ = "a branch of a complex log"



Complex chain rule: f, g analytic.

Then $h(z) = f(g(z))$ is analytic and

$$h'(z) = f'(g(z))\, g'(z).$$

Pf: Same as $\mathbb{R} \to \mathbb{R}$ proof. ($|z|$ = Modulus)

Fact: $\dfrac{d}{dz}(\log z) = \dfrac{1}{z}$

Why: Polar C-R Eqns $\Longrightarrow$ log is analytic.

Let $E(z) = e^z$. We know $E'(z) = E(z)$.

$$e^{\log z} = z$$

$$E(\log z) = z \leftarrow \text{Do chain rule}$$

$$\underbrace{E'(\log z)}\, \frac{d}{dz}(\log z) = \frac{d}{dz}(z) = 1$$

$E(\underbrace{\log z}_{z})$ So $\frac{d}{dz}(\log z) = \frac{1}{z}$ ✓

Powers:

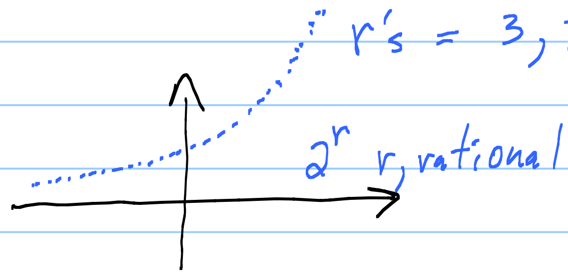
$2^3 = 2 \cdot 2 \cdot 2$

$2^{1/2} = x$ such $x^2 = 2$.

$2^{5/7} = (2^5)^{1/7} = x$ such that $x^7 = 2^5$.

$2^\pi = ?$ $\quad \lim 2^{r_j}$

r's $= 3, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \frac{31415}{10{,}000}, \ldots$



Aha! $\quad 2^\pi = e^{\pi \ln 2}$ $\qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

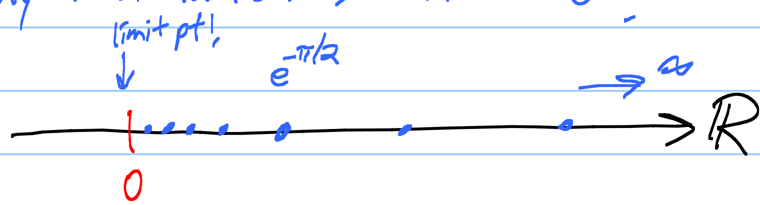
Complex case: $\quad a^b = e^{b \log a}$

EX: $\quad i^i = e^{i \log i} = e^{i(\underbrace{\ln|i|}_{0} + i(\frac{\pi}{2} + n 2\pi))}$

$= e^{-(\frac{\pi}{2} + n2\pi)}$ $\qquad n = 0, \pm 1, \pm 2$

$= e^{-\pi/2} \cdot e^{N 2\pi}$ $\quad (N = -n)$

∞ many real numbers! All > 0!



0 and ∞ are limits of values of i^i!

Gut feelings are still valid:

1) $z^n = e^{n\log z} = e^{n(\ln|z| + i\arg z)}$

$= e^{\ln|z|^n + in(\mathrm{Arg}\, z) + in2\pi}$ ← only one value

$e^{in2\pi} = 1, \; n \in \mathbb{Z}$

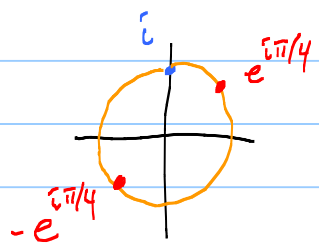
$z^n = \underbrace{z \cdot z \cdots z}_{n \text{ times}}$

2) Same for $z^{-n} = \left(\frac{1}{z}\right)^n$ ← one value.

3) $i^{1/2} = e^{\frac{1}{2}\log i} = e^{\frac{1}{2}\left(\underbrace{\ln|i|}_{0} + i\left(\frac{\pi}{2} + n2\pi\right)\right)}$

$= e^{i\frac{\pi}{4}} \underbrace{e^{in\pi}}_{\pm 1}$

only two values: $\pm\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$



4) $z^{p/q}$ ← has q values. (p, q no common factors)

5) z^w ← ∞ many values for generic z, w complex.

<u>Principal value</u>: $z^w = e^{w\,\mathrm{Log}\, z}$ ← Princ. Value Log

<u>Important convention</u>: $e^z = e^x e^{iy}$

$= e^x \cos y + i e^x \sin y$

<u>not</u> $e^z = e^{z\log e} = e^{z(\underbrace{\ln e}_{1} + in2\pi)}$

Convention: $2^z = e^{z \ln 2}$

$x^z = e^{z \ln x}$ when x real $x > 0$.

Chain rule: $h(z) = f(g(z))$

$w_0 = g(z_0)$

$$DQ = \frac{f(w) - f(w_0)}{w - w_0} = f'(w_0) + E(w)$$

mult by $w - w_0$

where $E(w) \to 0$ as $w \to w_0$.

Define $E(w_0) = 0$.

$$\boxed{f(w) - f(w_0) = f'(w_0)(w - w_0) + E(w)(w - w_0)}$$

Aha! If I let $w \to w_0$, see $f(w) \to f(w_0)$.

Fact: f complex diff'ble at a, then f is continuous at a.

Red box: let $w = g(z)$, $w_0 = g(z_0)$.

Then divide by $z - z_0$.

$$DQ_h = \frac{f(g(z)) - f(g(z_0))}{z - z_0} = f'(w_0)\left(\frac{g(z) - g(z_0)}{z - z_0}\right) + E(g(z))\left(\frac{g(z) - g(z_0)}{z - z_0}\right)$$

Let $z \to z_0$ $\quad DQ \longrightarrow f'(w_0)\, g'(z_0) + \underbrace{E(w_0)}_{0} \cdot g'(z_0)$

because g diff'ble at $z_0 \Rightarrow g$ continuous at z_0

So $g(z) \to g(z_0) = w_0$ as $z \to z_0$

So $E(g(z)) \to 0$ as $z \to z_0$.

EX: $e^{iz} = E(iz)$.

$$\frac{d}{dz} e^{iz} = \frac{d}{dz}\left(E(iz)\right) = E'(iz)\frac{d}{dz}(iz)$$
$$= E(iz) \cdot i$$
$$= ie^{iz}$$