Lesson 2 on 9.8, 9.9 Div, Curl

WebEx Off. Hr.
Thurs. 8-8:30pm Eastern Time

Div, grad, curl and all that. Schey.

$$\nabla f = \text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \leftarrow \text{vector field}$$

General vector field

$$\vec{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

Def^n: $\text{Div } \vec{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \leftarrow$ function

EX: $\vec{V} = e^{x \sin y}\hat{i} + xy^2\hat{j} + (x+z)\hat{k}$

$$\text{Div } \vec{V} = e^{x \sin y}(\sin y) + 2xy + 1$$

Notation: $\text{Div } \vec{V} = \nabla \cdot \vec{V}$

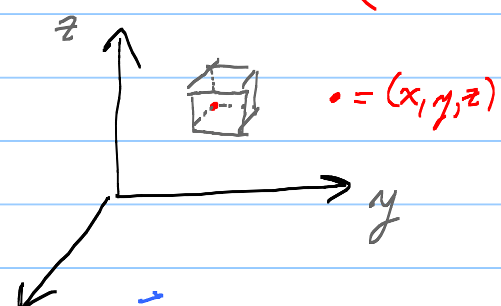
Laplacian: $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

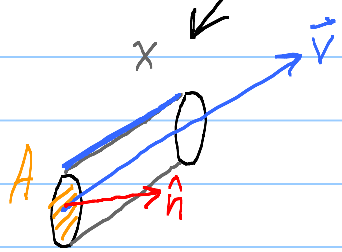
$$= \text{Div}(\nabla f) = \nabla \cdot \nabla f = \nabla^2 f$$

TeX nabla

Fluid flow: $\vec{V}$ = velocity field of a fluid

ρ = density $\leftarrow$ assume constant. (incompressible)

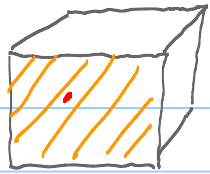




In time Δt, a tube gets pushed through.

Height tube $= (\vec{V}\Delta t) \cdot \hat{n}$

Volume $= A\, \vec{V} \cdot \hat{n}\, \Delta t$



velocity coming out

$= v_1(x+\Delta x, y, z) \quad = \vec{v}\cdot\hat{i}$

Stuff out in Δt = $v_1(x+\Delta x, y, z)\underbrace{(\Delta y\,\Delta z)}_{A}\,\Delta t$

<u>Back face:</u> $-\,v_1(x,y,z)(\Delta y\,\Delta z)\Delta t$

Add up: $\dfrac{v_1(x+\Delta x, y, z) - v_1(x,y,z)}{\Delta x}\,\underbrace{\Delta x\,(\Delta y\,\Delta z)}_{\Delta vol}\,\Delta t$

$$\approx \frac{\partial v_1}{\partial x}(\Delta Vol)\,\Delta t$$

Similarly for other sides:

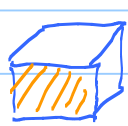
Get: Net outflow of fluid in volume in time Δt

$$\approx (\mathrm{Div}\,\vec{v})(\Delta vol)\,\Delta t$$

<u>Continuity Eqn:</u> $\mathrm{Div}\,\vec{V} = 0$ $\leftarrow$ Conservation of mass for incompressible fluid flow.

<u>Result:</u> If density ρ is <u>not</u> const.

$$\boxed{\mathrm{Div}\,(\rho\vec{v}) = -\frac{\partial\rho}{\partial t}}$$

<u>Why:</u>  <u>Mass</u> shoved out

$$\underbrace{\rho(x+\Delta x, y, z)\, v_1(x+\Delta x, y, z)}\,(\Delta x\, \Delta y)\, \Delta t$$

Get $\nabla(\rho \vec{v})$ where $\nabla \vec{v}$ before.

Next: Mass of box $= \rho\,(\Delta \text{vol})$

Net outflow in time Δt : $-\frac{\partial}{\partial t}\left[\rho(\Delta \text{vol})\right] \Delta t$

Last step: equate. Divide out $(\Delta \text{vol})\,\Delta t$.

$$\text{Curl } \vec{v} = \nabla \times \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{bmatrix}$$

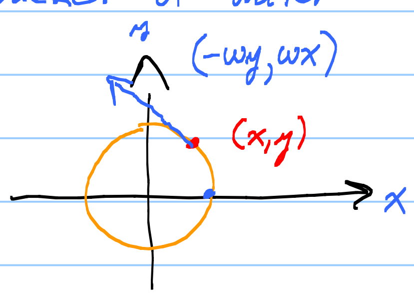
EX: $\vec{v} = e^{x \sin y}\, \hat{i} + x y^2\, \hat{j} + 0 \cdot \hat{k}$

$$\text{Curl } \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x \sin y} & x y^2 & 0 \end{bmatrix}$$

$$= 0\,\hat{i} + 0\,\hat{j} + \left(\frac{\partial}{\partial x}(x y^2) - \frac{\partial}{\partial y}(e^{x \sin y}) \right) \hat{k}$$

$$= \left(y^2 - x \cos y\, e^{x \sin y} \right) \hat{k}$$

EX: Spin a bucket of water

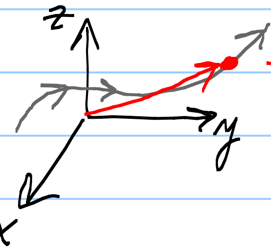


$$\vec{v} = -wy\, \hat{i} + wx\, \hat{j} + 0 \cdot \hat{k}$$

$$\text{curl } \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -wy & wx & 0 \end{bmatrix} = 2w\, \hat{k}$$

Think: Curl detects spin. Spits vector in orthog dir via right hand rule.

Motion of a particle:



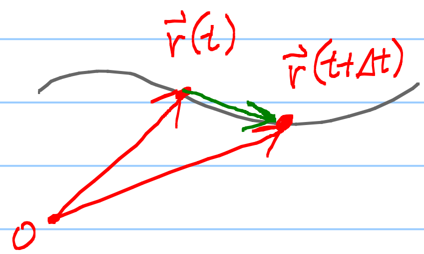
$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$= \vec{r}\,'(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j} + \dot{z}(t)\hat{k}$$

Big problem: Given a velocity field $\vec{V}$ of a fluid, find motion of a particle of fluid.

Look for $\vec{r}(t)$ such that $\vec{r}\,'(t) = \vec{V}$.
System of 3 ODE's.

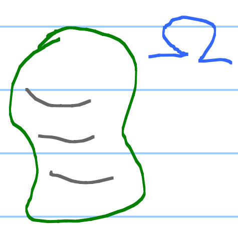


Fact: $\vec{v}(t) = \vec{r}\,'(t)$ points in tangential dir at $\vec{r}(t)$.

and $\|\vec{v}(t)\|$ = speed of particle

$$= \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

Chap 10: Divergence Thm.



$$\underbrace{\iiint_{\Omega} (\mathrm{Div}\,\vec{v})\, d(\mathrm{vol})}_{\text{fluid created}} = \underbrace{\iint_{\partial\Omega} \vec{v} \cdot d\vec{A}}_{\text{out flux}}$$