Lesson 21 on 14.2 Cauchy's Theorem

HWK 6: Lessons 18, 19, 20 due Wed.
WebEx Tues. 8-9 pm

Prob: Show that ze^{z^2} is analytic

Facts: 1) f, g analytic $\Rightarrow$ fg analytic & $(fg)' = f'g + fg'$

2) u, v C^1-smooth & satisfy C-R Eqns $\Rightarrow$ $u+iv$ analytic

3) Ω_1, Ω_2 domains $g: \Omega_1 \to \Omega_2$, $f: \Omega_2 \to \mathbb{C}$.
g analytic on Ω_1, f analytic on Ω_2.
Then $f \circ g$ is analytic on Ω_1. Chain Rule holds.

Step 1: $f(z) = z$ is analytic. $DQ = \frac{z-a}{z-a} = 1 \to 1$ as $z \to a$.

Step 2: $z^2 = z \cdot z$ is analytic by fact 1.

Step 3: e^z is analytic via C-R eqns.

Step 4: e^{z^2} is analytic by fact 3.

Step 5: $ze^{z^2} = z \cdot e^{z^2}$ is analytic by fact 1.

Product Rule: $h(z) = f(z)g(z)$

$$DQ = \frac{h(z) - h(a)}{z-a} = \frac{f(z)g(z) - f(a)g(a)}{z-a}$$

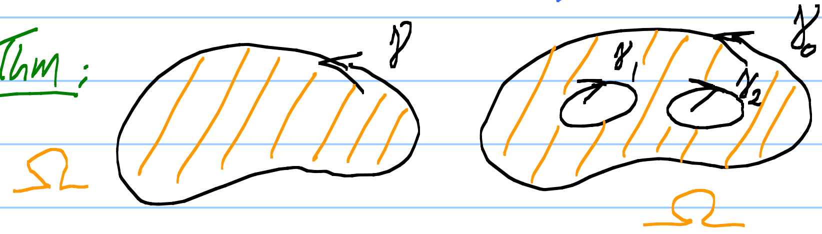
$$= \frac{[(f(z) - f(a)) + f(a)]\,g(z) - f(a)g(a)}{z-a}$$

$$= \frac{f(z) - f(a)}{z-a} \cdot g(z) + f(a)\,\frac{g(z) - g(a)}{z-a}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \text{as } z \to a.$$

$$f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

Green's Thm:



$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

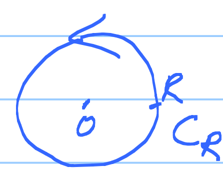
Cauchy's Thm: f analytic inside (Ω) and on γ, then $\int_\gamma f\,dz = 0$

Why: $\int_\gamma f\,dz = \int_\gamma [u+iv][dx+i\,dy]$

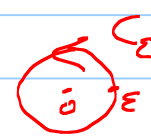
$$\overset{\text{Def}^n}{=} \left(\int_\gamma u\,dx - v\,dy\right) + i\left(\int_\gamma v\,dx + u\,dy\right)$$

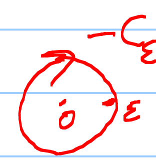
$$\underset{\text{Green's}}{=} \iint_\Omega \underbrace{-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{\substack{=0 \\ \text{C-R Eqn \#2}}}\,dA + i \iint_\Omega \underbrace{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}_{\substack{=0 \\ \text{C-R Eqn \#1}}}\,dA = 0$$

EX: $\int_{C_R} e^{z^4}\,dz = 0$



Recall: $\int_{C_\varepsilon} \frac{1}{z}\,dz = 2\pi i$



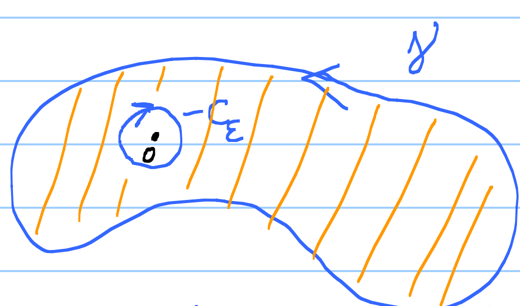


$z(t) = \varepsilon e^{it}$, $0 \le t \le 2\pi$

$z(t) = \varepsilon e^{-it}$, $0 \le t \le 2\pi$

Fact: $\int_\gamma f\,dz = -\int_{-\gamma} f\,dz$ by Riemann Sums.

EX:



Cauchy: $\left(\int_\gamma + \int_{-C_\varepsilon}\right) \frac{1}{z}\,dz = 0$

So $\int_\gamma \frac{1}{z}\,dz = -\int_{-C_\varepsilon} \frac{1}{z}\,dz = \int_{C_\varepsilon} \frac{1}{z}\,dz = 2\pi i$

Fact: If γ goes around origin once counterclockwise, then $\int_\gamma \frac{1}{z}\,dz = 2\pi i$.

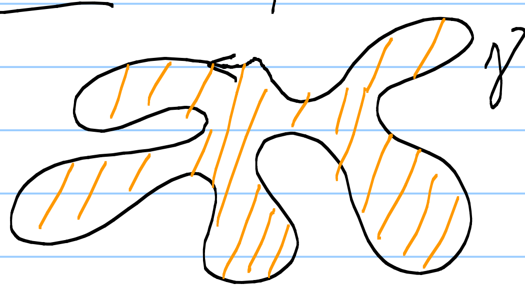
Hmmm: Why doesn't Cauchy Thm show $\int_\gamma \frac{1}{z}dz = 0$?

$\frac{1}{z}$ blows up at $z=0$. $\frac{1}{z}$ is not analytic inside γ.

But $\int_\gamma \frac{1}{z^2}\,dz = 0$. $\leftarrow \int_\gamma \frac{d}{dz}\left[\frac{-1}{z}\right] dz$

$= \left[\frac{-1}{z}\right]_{START}^{END} = 0$

Def^n: Simple closed curve:



$\leftarrow$ no self crossings.

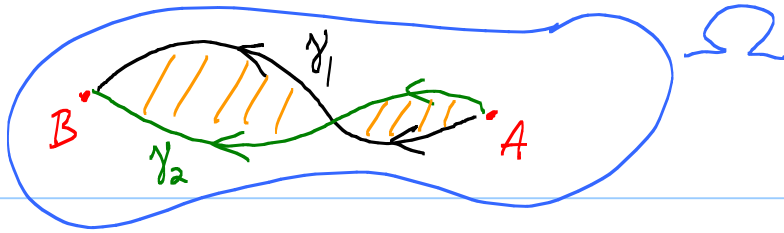
$\leftarrow$ closed.

Jordan curve Theorem: γ has an inside and an outside

Cauchy Thm 1: If f is analytic inside and on a simple closed curve γ, then $\int_\gamma f\,dz = 0$.

Cauchy Thm 2: If f is analytic on a simply connected domain, then $\int_\gamma f\,dz$ is I.O.P.

Cauchy 1 $\Rightarrow$ Cauchy 2.



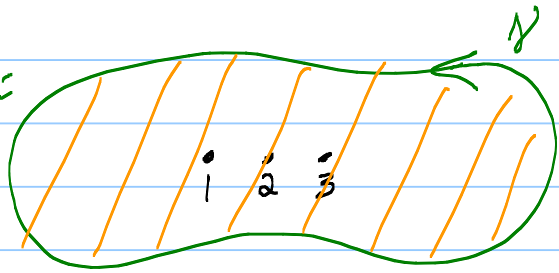
$$\left(\int_{\gamma_1} - \int_{\gamma_2}\right) f\,dz = \left(\int_{\gamma_1} + \int_{-\gamma_2}\right) f\,dz$$

$$= \left(\int_{\text{loop }1} + \int_{\text{loop }2}\right) f\,dz$$

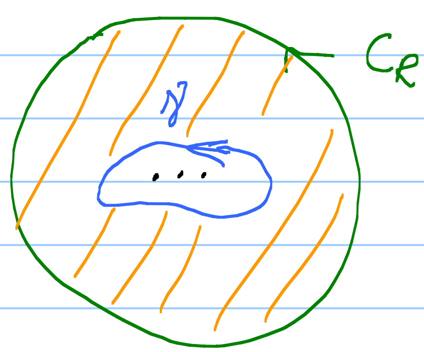
$$= 0 + 0 \quad \text{by Cauchy 1.}$$

Cor: If f is analytic on a simply connected domain, then $\int_\gamma f\,dz = 0$ for any closed curve γ in the domain.

EX:



$$\int_\gamma \underbrace{\frac{z+5}{(z-1)(z-2)(z-3)}}_{f(z)}\,dz = 0$$



Cauchy Thm:

$$\left(\int_{C_R} + \int_{-\gamma}\right) f(z)\,dz = 0$$

$$\int_{C_R} f\,dz = \int_\gamma f\,dz$$

$$\int_\gamma = -\int_{-\gamma}$$

Numerator estimate: $|z+5| \le |z| + 5$

z on C_R, then $|z| = R$. So $|z+5| \le R+5$ on C_R.

Denominator estimate: $|z-2| \geq \underbrace{\big||z|-2\big|}_{|R-2|}$

z on C_R, then $|z|=R$ $\qquad = R-2$ if $R>2$.

So, if $R>3$, then

Basic Estimate

$$\left|\int_{C_R} \frac{z+5}{(z-1)(z-2)(z-3)}\,dz\right| \leq \left(\max_{C_R}|f|\right) \text{Length}(C_R)$$

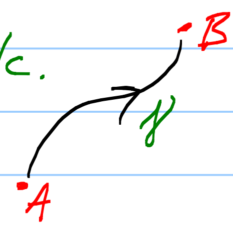
$$\leq \frac{R+5}{(R-1)(R-2)(R-3)} \cdot (2\pi R)$$

$$\longrightarrow 0 \text{ as } R \to \infty.$$

Hmmm: $\int_\gamma f\,dz = \int_{C_R} f\,dz \longrightarrow 0$ as $R \to \infty$.

Conclude that $\int_\gamma f\,dz = 0$.

Antiderivatives: Fund Thm Calc.



$$\int_\gamma \underbrace{f'}_{F}\,dz = f(B) - f(A)$$

$B=z$ moves; hold A fixed. $f(A) = $ const.

Hmmm: Look for an antiderivative of f, i.e., an analytic F such tha $f' = F$.

Big idea: Try $f(z) = \int_{\gamma_A^z} F(w)\,dw$.

I.O.P.!

This gives a well defined $f(z)$ on a simply connected domain.

Can show $f'(z) = F(z)$.

Fact: Analytic fcns have analytic antiderivatives on simply conn. domains.