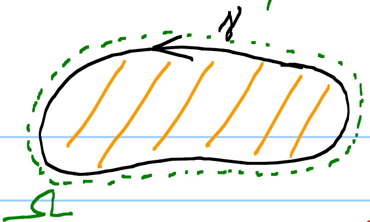
Lesson 22 on 14.3 the Cauchy Integral Formula

HWK 6: 18, 19, 20 due tonight

Cauchy Theorem:



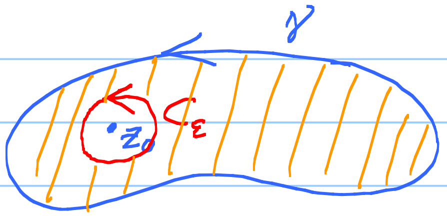
$$\int_\gamma f\, dz = 0$$

f analytic "inside and on γ" meaning there is an open set Ω containing the whole picture where f is analytic

simple closed curve γ

or $\int_\gamma f\, dz = 0$ if γ is a closed curve in a simply connected domain where f is analytic.

Last time:



Cauchy Thm with a hole

$$\left(\int_\gamma + \int_{-C_\varepsilon}\right)\frac{1}{z-z_0}\,dz = 0$$

$$\int_\gamma \frac{1}{z-z_0}\,dz = -\int_{-C_\varepsilon} = \int_{C_\varepsilon}\frac{1}{z-z_0}\,dz$$

C_ε: $z(t) = z_0 + \varepsilon e^{it}$ $\quad 0 \le t \le 2\pi$

$z'(t) = \varepsilon i e^{it}$

$$\int_{C_\varepsilon} = \int_0^{2\pi}\frac{1}{(\cancel{z_0}+\underline{\varepsilon e^{it}})-\cancel{z_0}}\left[\underline{\varepsilon}\, i\, \underline{e^{it}}\, dt\right]$$

$$= \int_0^{2\pi} i\, dt = 2\pi i$$

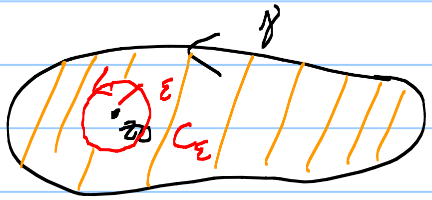
So $\boxed{\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i}$

Big idea: Replace f by a DQ for f in

this calculation.

$$DQ = \frac{f(z)-f(z_0)}{z-z_0} \longrightarrow f'(z_0) \quad \text{as } z \to z_0$$

So DQ is nice and bounded near z_0. There is an $M>0$ such that $|DQ| \le M$ if $|z-z_0|<\rho$.



Cauchy's:

$$\left(\int_\gamma + \int_{-C_\varepsilon}\right) DQ\, dz = 0$$

$$\int_\gamma DQ\, dz = \underbrace{\int_\gamma \frac{f(z)}{z-z_0}\, dz}_{\text{Cauchy integral}} - \underbrace{\int_\gamma \frac{f(z_0)}{z-z_0}\, dz}_{f(z_0) \underbrace{\int_\gamma \frac{1}{z-z_0}\, dz}_{2\pi i}}$$

$= 2\pi i\, f(z_0)$

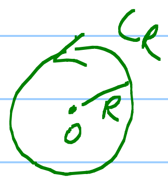
Aha! Get C.I. Formula if $\int_{C_\varepsilon} DQ\, dz \to 0$ as $\varepsilon \to 0$.

It does! $$\left|\int_{C_\varepsilon} DQ\, dz\right| \le \left(\underbrace{\max_{C_\varepsilon} |DQ|}_{\le M}\right) \underbrace{\text{Length}(C_\varepsilon)}_{2\pi\varepsilon}$$

Suppose $\varepsilon < \rho$.

$$\le 2\pi M \varepsilon \to 0 \quad \text{as } \varepsilon \to 0\,!$$

EX:



$$\int_{C_R} \frac{\overset{f(z)}{\boxed{e^z}}}{z-\underset{z_0}{1}}\, dz \overset{?}{=} 2\pi i\, f(z_0) = 2\pi i\, e^1$$

$$\int_{C_R} = \begin{cases} 0 & \text{if } R<1 \\ \text{not defined} & \text{if } R=1 \\ 2\pi e i & \text{if } R>1 \end{cases}$$

EX: $\displaystyle\int_{C_2} \frac{e^z}{z^2+1}\,dz$

Big idea. Partial Fractions.

$$\frac{1}{z^2+1} = \frac{1}{z^2-(i^2)} = \boxed{\frac{1}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i}}$$

Multiply $\boxed{}$ by denominator $(z-i)(z+i)$ =

$$1 = A(z+i) + B(z-i)$$

$$0\cdot z + 1 = \underbrace{(A+B)}_{=0}z + \underbrace{(Ai-Bi)}_{=1}$$

a) $A+B=0$ ← $\underline{B=-A}$

b) $Ai-Bi=1$ ← $(A-B)i=1$

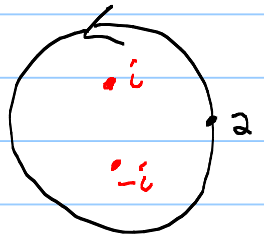
$[A-(-A)]i=1$

$\boxed{A=1/2i} = -\frac{1}{2}i$

$\boxed{B=-A=\frac{1}{2}i}$

So $\displaystyle\frac{1}{z^2+1} = \frac{-\frac{i}{2}}{z-i} + \frac{\frac{i}{2}}{z+i}$ and

$$\int_{C_2}\frac{e^z}{z^2+1}\,dz = -\frac{i}{2}\int_{C_2}\frac{\overset{f(z)}{\boxed{e^z}}}{z-\underset{z_0}{i}}\,dz + \frac{i}{2}\int_{C_2}\frac{\boxed{e^z}}{z-\underset{z_0=-i}{(-i)}}\,dz$$



$$= -\frac{i}{2}\left[\underbrace{2\pi i\, e^{i}}_{2\pi i\, f(z_0)}\right] + \frac{i}{2}\left[2\pi i\, e^{-i}\right]$$

$$= \pi e^{i} - \pi e^{-i}$$

$$= \pi (2i) \underbrace{\frac{e^{i} - e^{-i}}{2i}}_{\sin 1}$$

$$= 2\pi i \sin 1$$

Hmmm. If you know f on γ, it is completely determined inside γ!

Mind blowing consequence of the C.I. Formula

If f is analytic, then f is infinitely complex diff'ble! [u and v are C^∞ smooth!]

Why: $$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw$$

$$DQ = \frac{f(z) - f(a)}{z - a} = \frac{\left(\frac{1}{2\pi i}\right)}{z-a} \int_\gamma f(w) \underbrace{\left[\frac{1}{w-z} - \frac{1}{w-a}\right]}_{\frac{(w-a)-(w-z)}{(w-z)(w-a)}} dw$$

will cancel! ← $z-a$ on top →

$$= \frac{1}{2\pi i} \int_\gamma f(w) \frac{1}{(w-z)(w-a)}\, dw$$

$$\longrightarrow \frac{1}{2\pi i} \int_\gamma f(w) \frac{1}{(w-a)^2}\, dw \quad \text{as } z \to a.$$

This shows we can diff under $\int_\gamma$.

$$f'(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^2}\, dw$$

Can do it again and again!

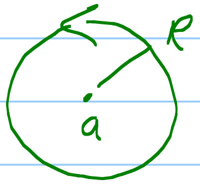
$$f''(a) = \frac{1}{2\pi i} \int_\gamma \underset{\substack{\uparrow \\ \text{derivative} \\ \text{w.r.t. } a}}{(-2)\frac{1}{(w-a)^3}} \underset{\substack{\uparrow \\ \frac{d}{da}(w-a)}}{(-1)} f(w)\, dw$$

$$f''(a) = \frac{2}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^3}\, dw$$

In general:

$$\boxed{f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\, dw}$$

Cauchy Estimates:

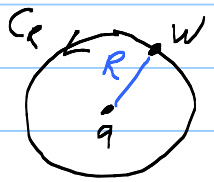


$$\left| f^{(n)}(a) \right| \le \frac{M\, n!}{R^n} \quad \text{where } M = \max_{C_R} |f|$$

Why:

$$\left| f^n(a) \right| \le \left| \frac{n!}{2\pi i} \int_{C_R} \frac{f(w)}{(w-a)^{n+1}}\, dw \right|$$

$$\left| \frac{f(w)}{(w-a)^{n+1}} \right| = \frac{|f(w)|}{\underbrace{|w-a|^{n+1}}_{R^{n+1}}}$$



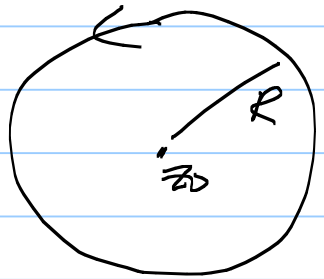
$$\le \frac{n!}{2\pi} \left(\max_{C_R} \frac{|f|}{R^{n+1}} \right) \underbrace{(2\pi R)}_{\text{Length}(C_R)}$$

$$\le \frac{n!\, M}{R^n} \quad \checkmark$$

Liouville's Thm: A bounded entire function must be constant.

Why: Say $\underbrace{f \text{ is analytic on } \mathbb{C}}_{\text{entire}}$

and $\underbrace{|f(z)| \le M \text{ for all } z \in \mathbb{C}.}_{\text{bounded on } \mathbb{C}}$



Do Cauchy Est for $f'(z_0)$ and let $R \to \infty$.