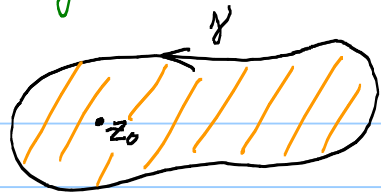
Lesson 23 on 14.4 Higher order Cauchy Integral Formulas HWK 7: 21, 22, 23

Cauchy Integral:



$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz$$

Higher order:

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz$$

$$f'(w) = \frac{1}{2\pi i}\int_\gamma f(z)\,\frac{d}{dw}\left[\frac{1}{z-w}\right]dz$$

$$= \frac{1}{2\pi i}\int_\gamma f(z)\,\frac{(-1)}{(z-w)^2}\,\underbrace{(-1)}_{\frac{d}{dw}(z-w)}\,dz$$

Repeat!

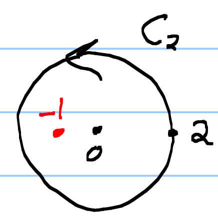
$$\boxed{f^{(n)}(w) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^{n+1}}\,dz}$$

EX:

$$\int_{C_2} \frac{z^{100}}{(z+1)^3}\,dz$$

$f(z) = z^{100}$

$(z+1)^3 = (z-(-1))^3$, $3 = n+1$, $\boxed{n=2}$, $w=-1$



$$= \frac{2\pi i}{n!} f^{(n)}(w) = \frac{2\pi i}{2!} f''(-1)$$

$$= \pi i \;100\cdot 99\, z^{98}\Big|_{z=-1}$$

$$= 9900\pi i$$

EX:

$$\int_{C_1} \frac{\operatorname{Log} z}{z}\,dz \overset{?}{=} \int_{C_1} \frac{f(z)}{z-0}\,dz = 2\pi i\, f(0)$$

Oops! Log has singular

behavior inside C_1.

$C_1: \; z(t) = e^{it}$

$z'(t) = ie^{it}$

$\text{Log}\, e^{it} = \underbrace{\text{Ln}|e^{it}|}_{\text{Ln}1=0} + i\,\underbrace{\text{Arg}\, e^{it}}_{t}$

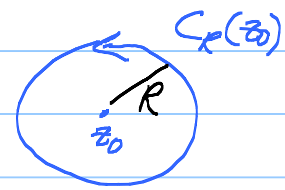
Aha! $-\pi < t \leq \pi$ instead of the usual

$$\int_{C_1} \frac{\text{Log}\, z}{z}\, dz = \int_{-\pi}^{\pi} \frac{it}{e^{it}} \left[ie^{it}\, dt\right]$$

$$= \int_{-\pi}^{\pi} -t\, dt = 0$$

Cauchy Estimates: $\gamma = C_R(z_0)$

$z_0 = $ Center.

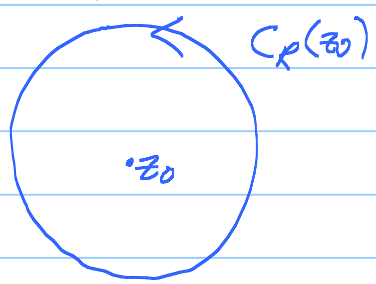


Basic estimate on Higher order Cauchy yields

$$\left|f^{n}(z_0)\right| \leq \frac{n!\,M}{R^n} \quad \text{where } M = \max_{C_R(z_0)} |f|$$

Liouville's Theorem: Bounded entire fcns must be constant.

Why:



Suppose f analytic on $\mathbb{C}$ and $|f(z)| \leq M$ for all z.

$$\left|f'(z_0)\right| \leq \frac{1!\,M}{R^1} \rightarrow 0 \text{ as } R \rightarrow \infty \, !$$

So $f'(z_0) = 0$. But z_0 is arbitrary!

So $f' \equiv 0$ on $\mathbb{C}$. Hence $f \equiv$ const on $\mathbb{C}$. ✓

Fund Thm Algebra: $P(z) = a_N z^N + \cdots + a_1 z + a_0$

with $N \geq 1$, $(a_N \neq 0)$. Then P has a root in $\mathbb{C}$, i.e., there is a z_0 with $P(z_0) = 0$.

Why: Suppose P has no complex root. Then $\frac{1}{P(z)}$ is entire!

Fact: $\lim_{|z| \to \infty} |P(z)| = \infty$. So $\left|\frac{1}{P(z)}\right| \to 0$ as $|z| \to \infty$, and $\frac{1}{P}$ is a bounded fcn. Liouville's $\Rightarrow$ $\frac{1}{P} = \text{const.}$

$\Rightarrow$ $P = \text{const.}$

But $|P| \to \infty$ as $|z| \to \infty$. $\lightning$ (contradiction!)

Fact: $$P(z) = z^N \Bigg[\underbrace{\overbrace{a_N + \frac{a_{N-1}}{z} + \cdots + \frac{a_0}{z^N}}^{H(z)}}_{\to\, a_N \text{ as } |z| \to \infty} \Bigg]$$

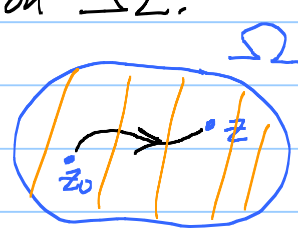
So $|H(z)| > \frac{|a_N|}{2}$ if $|z| > \underset{\substack{\uparrow \\ \text{some } R_0}}{R_0}$

$$|P(z)| = |z|^N |H(z)| > \frac{|a_N|}{2} |z|^N \text{ if } |z| > R_0.$$

$\to \infty$ as $|z| \to \infty$. ✓

Antiderivatives of analytic fcns: If f is analytic on a simply connected domain Ω, then f has an analytic antiderivative on Ω.

(no holes in $\mathbb{R}^2$)



Why: Suppose $F' = f$ on Ω.

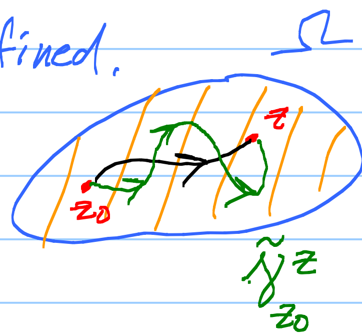
$$\int_{\gamma_{z_0}^{z}} \underbrace{F'}_{f}\, dw = F(z) - F(z_0)$$

$$F(z) = \underbrace{F(z_0)}_{\text{const.}} + \int_{\gamma_{z_0}^{z}} f(w)\,dw$$

Aha! If I don't have F, try to define

$$\boxed{F(z) = \int_{\gamma_{z_0}^{z}} f(w)\,dw}$$

Step 1: F is well defined.

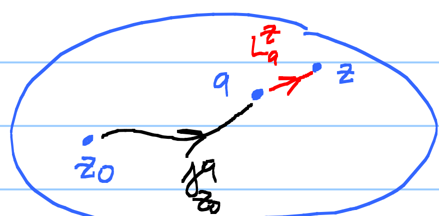


Cauchy's Thm $\Longrightarrow$

$\int$ is I.O.P.

($\gamma_{z_0}^{z} \cup (-\tilde{\gamma}_{z_0}^{z})$ is a closed path in a s.c. domain)

Step 2: $DQ \to f$



L_a^z line!

$$\frac{F(z)-F(a)}{z-a} = \frac{1}{z-a}\Bigg[\underbrace{\left(\int_{\gamma_{z_0}^{a}} f\,dw + \int_{L_a^z} f\,dw\right)}_{F(z)} - \underbrace{\int_{\gamma_{z_0}^{a}} f\,dw}_{F(a)}\Bigg]$$

$$= \frac{1}{z-a}\int_{L_a^z} f\,dw \qquad \left[\begin{array}{l} L_a^z : \ \zeta(t) = a + t(z-a) \\ \qquad 0 \le t \le 1 \\ \zeta'(t) = z-a \end{array}\right.$$

$$DQ = \frac{1}{z-a}\int_0^1 f\big(a+t(z-a)\big)\,\big[\underline{(z-a)}\,dt\big]$$

cancels!

$$= \int_0^1 f(a + t(z-a))\, dt$$

$$\longrightarrow \underbrace{\int_0^1 f(a)\, dt}_{= f(a)} \quad \text{as } z \to a$$

<u>Morera's Theorem</u>: If f is a <u>continuous</u> complex valued fcn on a simply connected domain Ω and

$$\int_\gamma f\, dz = 0$$

for every closed curve in Ω, then f is <u>analytic</u>.

<u>Why</u>: $F(z) = \int_{\gamma_{z_0}^{z}} f\, dw$ would be <u>well</u> defined. Same argument as above shows $F' = f$. Aha! F is analytic. Higher order Cauchy integrals show that F' is analytic too. But $F' = f$. So f is analytic!