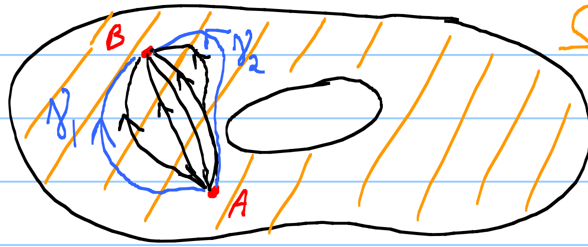
Lesson 24 on 15.1 Complex sequences & series

HWK 7: Lessons 21, 22, 23 due Wed.,
WebEx Office Hour Tues. 8–9 pm

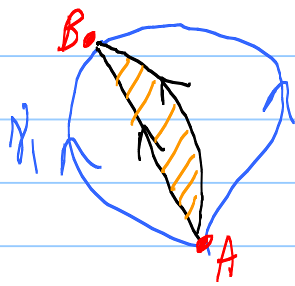
Deformation of paths
p. 656



Ω $\quad$ f analytic on Ω

$$\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$$

Why:



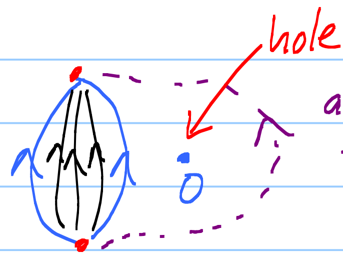
Cauchy's Thm

$$\left(\int_{\gamma_{out}} + \int_{\gamma_{back}}\right) f\,dz = 0$$

Also true for closed curves.

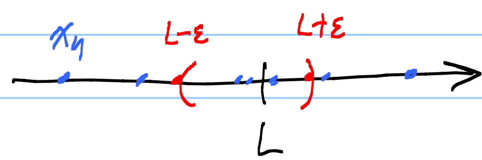
$$\int_\gamma = \int_C$$

Hole: $\displaystyle\int_\gamma \frac{1}{z}\,dz$



adds $2\pi i$ to $\int$!

Sequences: $\mathbb{R}$



$\lim_{n\to\infty} x_n = L$ means: Given an $\varepsilon > 0$, there is an N such that $|x_n - L| < \varepsilon$ for $n > N$.

$\mathbb{C}$: Words same! Picture different:

$|z_n - L| < \varepsilon$ if $n > N$.

$z_n \in D_\varepsilon(L)$ for $n > N$.

EX: $\lim_{n\to\infty} \frac{e^{in}}{n} = 0$

$$\left|\frac{e^{in}}{n}\right| = \left|\frac{e^{in}}{n} - 0\right| = \frac{|e^{in}|}{n} = \frac{1}{n} < \varepsilon$$

$$n > \frac{1}{\varepsilon}$$

Fact: $\lim_{n\to\infty} z_n = L \iff \begin{cases} \lim_{n\to\infty} x_n = \mathrm{Re}\, L \\ \lim_{n\to\infty} y_n = \mathrm{Im}\, L \end{cases}$

$z_n = x_n + i y_n$

Complex Series: $\sum_{n=1}^{\infty} z_n$. $S_N = \sum_{n=1}^{N} z_n$

Series converges $\iff$ S_N converge.

Fact: If $\sum_0^\infty z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

Why: $a_N = S_N - S_{N-1}$

$\downarrow \quad \downarrow$ as $N \to \infty$.

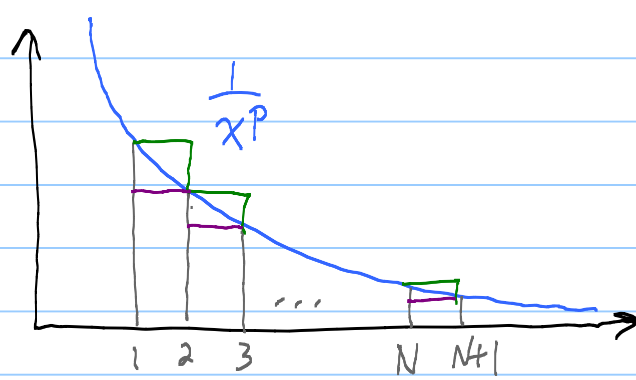
$L - L = 0$

One way implication! $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

even though terms $\to 0$.

Divergence test #1: If z_n do not tend to zero, then $\sum_1^\infty z_n$ does not converge.

Integral Test:



$$\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{(N+1)^p} < \int_1^{N+1} \frac{1}{x^p}\,dx < \frac{1}{1^p}+\cdots+\frac{1}{N^p}$$

See $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{Converges if } p>1 \\ \text{Diverges if } 0<p\le 1 \end{cases}$

<u>Defn</u>: $\displaystyle\sum_1^{\infty} z_n$ <u>converges absolutely</u> if

$\displaystyle\sum_1^{\infty} |z_n|$ converges $\left(\sum < \infty\right)$.

<u>Fact</u>: An absolutely convergent series converges.

<u>Why</u>: $\left.\begin{array}{l} |x_n| \le |z_n| \\ |y_n| \le |z_n| \end{array}\right\}$ so $\begin{array}{l} \sum |x_n| \le \sum |z_n| \\ \sum |y_n| \le \sum |z_n| \end{array}$

abs conv $\Rightarrow$ conv in $\mathbb{R}$. ✓

<u>Geometric Series</u>: $\displaystyle\sum_{n=0}^{\infty} z^n$

$$\begin{aligned} S_N &= 1+z+z^2+\cdots+z^N \\ -\quad zS_N &= \quad\; z+z^2+\cdots+z^N+z^{N+1} \\ \hline S_N - zS_N &= 1 - z^{N+1} \end{aligned}$$

$$S_N = \frac{1-z^{N+1}}{1-z} = \frac{1}{1-z} - \frac{z^{N+1}}{1-z}$$

see conv properties

<u>Case $|z|<1$</u>: $|z^{N+1}| = |z|^{N+1}$

$\to 0$ as $N\to\infty$.

See $\displaystyle\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

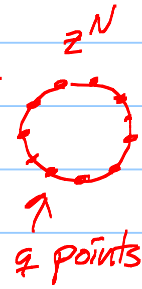
<u>Case $|z|\ge 1$</u>: $|z^n| = |z|^n$ ← does not $\to 0$ as $n\to\infty$.

Div. Test #1 says series diverges.

$|z| = 1$ case is odd! $\quad z = 1$: $\quad 1+1+1+1 \cdots \to \infty$

$z = -1 \quad 1-1+1-1+\cdots \quad 1, 0, 1, 0, \ldots$

Error term $\quad \dfrac{z^{N+1}}{1-z} \leftarrow z = e^{i p\pi/q}$



Good area is an open circle. $D_1(0)$.

Radius of convergence! $\quad R = 1$

L'Hôpital's Rule holds in $\mathbb{C}$:

$\lim_{z \to a} f(z), g(z) = 0$. Then

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)} \leftarrow \text{if } g'(a) \neq 0.$$

Why: $$\frac{f(z)}{g(z)} = \frac{f(z) - 0}{g(z) - 0} = \frac{\left(\dfrac{f(z) - f(a)}{z - a}\right)}{\left(\dfrac{g(z) - g(a)}{z - a}\right)}$$

$$\to \frac{f'(a)}{g'(a)} \quad \text{as } z \to a \text{ provided that } g'(a) \neq 0.$$

Comparison Tests: Suppose $r_n > 0$. $\sum_{n=1}^{\infty} r_n < \infty$ and $|z_n| \le r_n$, then $\sum_{n=1}^{\infty} z_n$ converges (absolutely)

Ratio Test: (Compare to a geometric series.)

Suppose $\sum_{n=1}^{\infty} z_n$ is a complex series and suppose further

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| \text{ exists and } = L.$$

Then: If $L<1$, series converges absolutely.

If $L>1$, terms don't $\to 0$, so series diverges.

If $L=1$, the test fails.

$\left(\sum \frac{1}{n^p} \leftarrow L=1.\quad 0<p\le 1 \text{ div.}\quad p>1 \text{ conv.}\right)$

Why: Say $L<1$. $\left|\frac{z_{n+1}}{z_n}\right| \to L < \rho < 1$

pick a ρ like this

So $\exists N$ such that $\left|\frac{z_{n+1}}{z_n}\right| < \rho$

if $n \ge N$.

$|z_{N+1}| < \rho\,|z_N| \qquad n=N$

$|z_{N+2}| < \rho\,|z_{N+1}| < \rho^2 |z_N| \qquad n=N+1$

$|z_{N+3}| < \rho^3 |z_N| \qquad n=N+2$

Compare tail end $\sum_{n=N}^{\infty} z_n$ to geometric series $\sum_{n=0}^{\infty} |z_N|\,\rho^n$, which converges absolutely!

EX: $\sum_{n=1}^{\infty} \underbrace{3^n\, n\, z^n}_{a_n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}(n+1)\,z^{n+1}}{3^n\, n\, z^n}\right| = 3\left(\frac{n+1}{n}\right)|z|$$

$$= 3\left(1+\frac{1}{n}\right)|z| \xrightarrow[n\to\infty]{} \underbrace{3|z|}_{L} \quad \begin{matrix} <1 & \text{conv} \\ >1 & \text{div.} \end{matrix}$$

$$\begin{cases} |z| < \frac{1}{3} & \text{conv.} \\ |z| > \frac{1}{3} & \text{div} \end{cases} \qquad R = \frac{1}{3}.$$