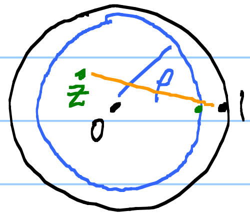
Lesson 25 on 15.2 Power Series

HWK 7: 21, 22, 23 due tonight
Exam 2 on Tues, March 22
8-9 pm in EE 129.

$$\underbrace{1+z+z^2+\cdots+z^N}_{\sum_{n=0}^{N} z^n} = \underset{\text{Limit}}{\frac{1}{1-z}} - \underbrace{\frac{z^{N+1}}{1-z}}_{\substack{\text{ERROR} \\ E_N(z)}}$$

ERROR $\to 0$ when $|z|<1$. R. of C. $= 1$



$0<\rho<1 \qquad z \in D_\rho(0)$.

$$|E_N(z)| = \frac{|z|^{N+1}}{|1-z|} \le \frac{\rho^{N+1}}{|1-z|} \le \frac{\rho^{N+1}}{1-\rho}$$

$\to 0$ as $N \to \infty$.

Fact: Estimate does not depend on $z \in D_\rho(0)$.
We get uniform convergence.

EX: $f(z) = \sum_{n=0}^{\infty} \underbrace{\frac{z^n}{n!}}_{u_n} \leftarrow e^z$!

Ratio Test : $\left|\frac{u_{n+1}}{u_n}\right| = \frac{\left|\frac{z^{n+1}}{(n+1)!}\right|}{\left|\frac{z^n}{n!}\right|} = \frac{|z|}{n+1}$

$\xrightarrow[n\to\infty]{} 0 = L < 1$

$L<1$: Series converges for all z.

$R =$ R. of C. is infinite. $R = \infty$.

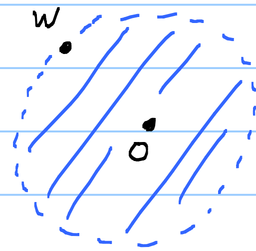
EX: $\sum_{n=0}^{\infty} n!\, z^n \qquad \left|\frac{u_{n+1}}{u_n}\right| \to (n+1)|z| \to \infty$ as $n \to \infty$ if $|z| \neq 0$.

Series diverges for $z \neq 0$. Only converges at $z=0$.

Say $R=0$.

<u>Fact</u>: If $\sum_{n=0}^{\infty} a_n z^n$ converges at $w \neq 0$,

then it converges in $\{z : |z| < |w|\}$.



<u>Fact, part 2</u>: If $\sum_{n=0}^{\infty} a_n z^n$ diverges at w,

then it diverges for all z with $|z| > |w|$.

<u>Consequence</u>: $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ has a R. of C.

$|z-z_0| < R$: Converges

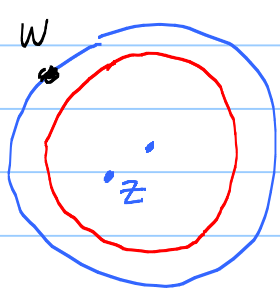
$|z-z_0| > R$: Diverges

$|z-z_0| = R$: Don't know, don't care.

<u>Why Fact true</u>: Suppose $\sum_0^{\infty} a_n w^n$ converges.

Then $a_n w^n \to 0$ as $n \to \infty$.

So $\exists N$ such that $|a_n w^n| < 1$ if $n > N$.



Look at $|a_n z^n|$

$$= \left| a_n w^n \cdot \frac{z^n}{w^n} \right|$$

$$= |a_n w^n| \cdot \left|\frac{z}{w}\right|^n$$

$$\le 1 \cdot \Big(\underbrace{\frac{|z|}{|w|}}_{\rho < 1}\Big)^n$$

if $n > N$.

Compare tail end of power series to convergent geometric series.

Fact: Estimate shows uniform convergence on $D_\rho(z_0)$ when $0<\rho<R$.

Famous Formula: Hadamard's Formula

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$ ← always works

Note: $\limsup_{n\to\infty} r_n = \lim_{N\to\infty} \sup\{r_n : n \geq N\}$

Book: Cauchy-Hadamard formula:

If $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists and is equal to L. Then $L =$ R. of C.

Warning: Don't use C-H Formula. Use Ratio Test directly.

EX: $\sum_{n=0}^{\infty} \underbrace{n\, 2^n z^{2n}}_{u_n} = a_2 z^2 + a_4 z^4 + a_6 z^6 + \cdots$

C-H Formula $\left|\frac{a_n}{a_{n+1}}\right|$ ← odd $n+1$ terms $=0$!

C-H bombs!

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)2^{n+1}\, z^{2(n+1)}}{n\, 2^n\, z^{2n}}\right| = \left(1+\tfrac{1}{n}\right) 2\,|z|^2$$

$$\xrightarrow[n\to\infty]{} 1\cdot 2\cdot |z|^2 \quad \begin{matrix} <1 & \text{conv.} \\ >1 & \text{div.} \end{matrix}$$

$$|z|^2 < \frac{1}{2} \quad \text{conv.}$$

$$|z| < \frac{1}{\sqrt{2}} \;.\; \leftarrow R = 1/\sqrt{2}$$

EX: $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{n!}{n^n} z^n}_{u_n}$

$0! = 1 \leftarrow$ convention

$0^0 \leftarrow$ not defined

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\frac{(n+1)!}{(n+1)^{n+1}} z^{n+1}}{\frac{n!}{n^n} z^n}\right| = \underbrace{\frac{(n+1)!}{n!}}_{n+1} \frac{n^n}{(n+1)^{n+1}} |z|$$

$$= \frac{\frac{1}{n^n} n^n}{\frac{1}{n^n} (n+1)^n} |z| = \frac{1}{(1+\frac{1}{n})^n} |z| \rightarrow \frac{1}{e}|z| \begin{array}{ll} <1 & \text{conv.} \\ >1 & \text{div} \end{array}$$

Famous limit: $\displaystyle\lim_{n\to\infty} \left(1+\frac{b}{n}\right)^n = e^b$

So $\underline{\underline{R = e}}$.

Why: $u_n = \left(1+\frac{b}{n}\right)^n$

Then $\underline{\ln u_n} = \ln\left(1+\frac{b}{n}\right)^n = n \ln\left(1+\frac{b}{n}\right)$

$$\rightarrow \infty \cdot 0$$

$$= \frac{\ln\left(1+\frac{b}{n}\right)}{\left(\frac{1}{n}\right)} \rightarrow \frac{0}{0}$$

$$\xrightarrow{L'H} \lim_{n\to\infty} \frac{\frac{1}{(1+\frac{b}{n})}\left(-\frac{b}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{b}{1+\frac{b}{n}} = b$$

Finally, $u_n = e^{\ln u_n} \longrightarrow e^b$ as $n \to \infty$

because e^x is continuous.

EX: $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{\ln n}{n^2}(z-2)^n}_{u_n}$

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{\frac{\ln(n+1)}{(n+1)^2}|z-2|^{n+1}}{\frac{\ln n}{n^2}|z-2|^n} = \underbrace{\frac{\ln(n+1)}{\ln n}}_{\downarrow \text{L'H } \frac{\infty}{\infty} \quad 1} \underbrace{\frac{n^2}{(n+1)^2}}_{\to 1} |z-2|$$

$$\to 1 \cdot 1 \cdot |z-2| \begin{array}{l} <1 \text{ conv} \\ >1 \text{ div.} \end{array}$$

So $R = 1$.

EX: $\displaystyle\sum_{n=0}^{\infty} \underbrace{n!\, z^{n!}}_{u_n}$ ← No C-H Formula!

Ratio Test: $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!\, z^{(n+1)!}}{n!\, z^{n!}}\right|$

$$= (n+1)\,|z|^{(n+1)!-n!} = (n+1)|z|^{n\,n!}$$

$$\left[\begin{aligned} (n+1)! - n! &= (n+1)n! - n! \\ &= n!\big[(n+1)-1\big] = n\,n! \end{aligned}\right]$$

Claim: $\to 0$ if $|z| < 1$. So $R = 1$.

$\to \infty$ if $|z| > 1$.