Lesson 26 on 15.3 Power Series, II

Practice Problems, old Exam 2 on
MA 528 Home Page

Hwk 8: 24, 25, 26 due Wed
after Spring Break 3/23.
Exam 2 Tues. March 22 8-9 pm
in EE 129.
Monday after break: Review Lec.

Why does $nr^n \to 0$ as $n \to \infty$
when $0 < r < 1$?

$r^n \to 0$ as $n \to \infty$ if $r < 1$.

$r^n < \varepsilon < 1$?

$\ln r^n < \ln \varepsilon$

$n \ln r < \ln \varepsilon$

$n > \frac{\ln \varepsilon}{\ln r}$ both <0

$n \cdot r^n \quad \infty \cdot 0$

$$= \frac{n}{r^{-n}} \xrightarrow{L'H} \lim_{n\to\infty} \frac{1}{(-\ln r)e^{-n\ln r}} = \lim_{n\to\infty} \frac{r^n}{-\ln r} = 0 \checkmark$$

$r^{-n} = e^{-n\ln r}$

Or Ratio Test shows $\sum_{n=0}^{\infty} n z^n$ has R.of.C $R=1$.

So terms $nr^n \to 0$ as $n \to \infty$ if $0 < r < 1$.

Big facts for today: 1) If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$
with Radius of Convergence (R.of C), $R > 0$ or $= \infty$,
then f is analytic on $D_R(z_0)$, and $f'(z)$
has a power series expansion $f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$
that has the same R.of C. R.

2) Taylor's Formula: $a_n = \frac{f^{(n)}(z_0)}{n!}$

3) If (Power Series A) = (Power Series B) then

$a_n = b_n$ for all n.

(Identity Theorem.)

Why 1: Hadamard's Formula shows that

$\underbrace{\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}}_{F(z)}$ has the same R. of C. as

$\sum_{0}^{\infty} a_n (z-z_0)^n$. [Reason: $\lim_{n\to\infty} \sqrt[n]{n} = 1$.]

Next step: $DQ - F = \frac{f(z)-f(a)}{z-a} - F(a)$

$\sum_{n=0}^{\infty} a_n \underbrace{\left[\frac{z^n - a^n}{z-a} - n a^{n-1}\right]}$ ← taking $z_0 = 0$ here

algebra $= (z-a)$ Poly

and Δ ineq. shows $|\quad| \le |z-a|$ (tail end of geom. series)

Why 2: $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$

$f(0) = a_0 + 0 + \cdots + 0 + \cdots = a_0$ ✓

$f'(z) = 0 + a_1 + 2a_2 z + \cdots + n a_n z^{n-1} + \cdots$

$f'(0) = a_1 + 0 + \cdots = a_1$ ✓

$f''(z) = 0 + 0 + 2\cdot 1\, a_2 + 3\cdot 2 a_3 z + \cdots$

$f''(0) = 2\cdot 1\, a_2 \qquad a_2 = \frac{f''(0)}{2!}$ ✓

etc.

EX: $E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$ $R = \infty$ (Ratio Test)

$E'(z) = 0 + 1 + \frac{2}{2!} z^1 + \frac{3}{3!} z^2 + \cdots + \frac{n}{n!} z^{n-1} + \cdots$

$= 1 + z + \frac{z^2}{2!} + \cdots + \frac{1}{(n-1)!} z^{n-1} + \cdots$

Aha!

$= E(z)$

Remark: Can "discover" $E(z)$ by

$$\left(\sum a_n z^n\right)' = \left(\sum a_n z^n\right), \quad a_0 = 1.$$

Big moment: $E(z) = e^z$.

Why: $$\frac{d}{dz}\left[\frac{E(z)}{e^z}\right] = \frac{E' e^z - E \frac{d}{dz}(e^z)}{(e^z)^2}$$

e^z non-vanishing

$$= \frac{E(z)e^z - E(z)e^z}{e^{2z}} \equiv 0.$$

So $\frac{E(z)}{e^z} = C$, a const., and $E(z) = Ce^z$.

Finally, take $z=0$ to see that $C=1$. ✓

EX: $$e^{iz} = 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots$$

$$= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \cdots$$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)$$

$$+ i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$

Repeat for e^{-iz}.

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = z - \frac{z^3}{3!} + \cdots$$

EX: Compute $\sum_{n=0}^{\infty} \frac{n}{2^n}$ and $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$.

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad R = 1$$

$$\left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2} = 0 + 1 + 2z + 3z^2 + 4z^3 + \cdots$$

$R=1$

Big idea! Multiply by z:

$$(*) \quad \frac{z}{(1-z)^2} = z + 2z^2 + \underline{\underline{3z^3}} + 4z^4 + \cdots$$

Aha! $z=\frac{1}{2}$ $\quad \frac{1/2}{(1-\frac{1}{2})^2} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots + \frac{n}{2^n} + \cdots$

Next, $(*)'$ $\quad \frac{d}{dz}\left[\frac{z}{(1-z)^2}\right] = 1 + 2\cdot 2z + \cdots + n^2 z^{n-1} + \cdots$ $\quad R=1$

Mult. by z: $\quad z\frac{d}{dz}\left[\frac{z}{(1-z)^2}\right] = \underbrace{2\cdot 2z^2 + 3\cdot 3z^3 + \cdots + n^2 z^n + \cdots}_{\sum_{n=0}^{\infty} n^2 z^n}$ $\quad R=1$

Finally, set $z=\frac{1}{2}$.

EX: $\cos^2 z = \frac{1}{2}(1 + \cos \underline{\underline{2z}})$

$$= \frac{1}{2}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2z)^{2n}\right]$$

EX: $f(z) = z^3$ centered at $z_0 = 1$.

$f(1) = 1 \quad \leftarrow a_0 = 1$

$f'(1) = 3\cdot 1^2 = 3 \quad \leftarrow a_1 = \frac{3}{1!} = 3$

$f''(1) = 3\cdot 2\cdot 1 = 6 \quad \leftarrow a_2 = \frac{6}{2!} = 3$

$f'''(1) = 6 \quad \leftarrow a_3 = \frac{6}{3!} = 1$

$f^{(n)}(1) = 0$ for $n \geq 4 \quad \leftarrow a_n = 0$ for $n \geq 4$

$$z^3 \stackrel{?}{=} 1 + 3(z-1) + 3(z-1)^2 + (z-1)^3$$ yes. ✓

EX: $f(z) = \operatorname{Log} z$, $z_0 = i$.

$$f(i) = \operatorname{Log} i = \underbrace{\operatorname{Ln}|i|}_{\operatorname{Ln}1=0} + i\underbrace{\operatorname{Arg} i}_{\frac{\pi}{2}} = i\frac{\pi}{2} \leftarrow a_0$$

$$f'(z) = \frac{1}{z} \qquad f'(i) = \frac{1}{i} = -i \leftarrow a_1$$

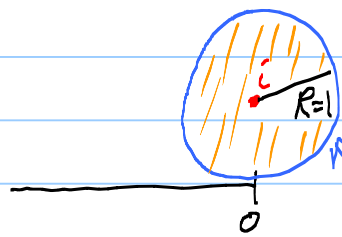
$$f''(z) = \frac{-1}{z^2} \qquad f''(i) = 1 \leftarrow a_2 = \frac{1}{2}$$

$$f'''(z) = \frac{(-1)(-2)}{z^3} \qquad f'''(i) = \frac{-2!}{i^3} = i\,2! \leftarrow a_3 = \frac{i}{3}$$

$$\vdots$$

$$\operatorname{Log} z \overset{?}{=} i\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-i^n}{n}(z-i)^n \qquad R = ?$$

$\leftarrow$ Ratio Test shows $R=1$



$\leftarrow$ Taylor's Series converges to Log z inside.

Biggest circle where Log z is analytic