2. (20) Find a harmonic conjugate of $\underbrace{y+e^x\cos y}_{u}$. Look for v with

$$\begin{cases} v_x = -u_y = -(1-e^x\sin y) = -1+e^x\sin y & (A) \\ v_y = u_x = e^x\cos y & (B) \end{cases}$$

Use (A): $v = \int(-1+e^x\sin y)\,dx = \underbrace{-x+e^x\sin y + h(y)}_{v}$!

Use (B): $\frac{\partial}{\partial y}\left[-x+e^x\sin y + h(y)\right] \overset{\text{want}}{=} e^x\cos y$

$$0 + \cancel{e^x\cos y} + h'(y) = \cancel{e^x\cos y}$$

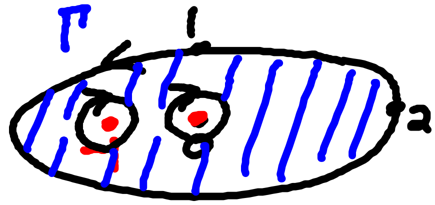
Need $h'(y)=0$.

So $h(y)=c$. $\boxed{v=-x+e^x\sin y + C}$ ok

Answer:

4. (20) i) Let Γ be the ellipse $x^2/4+y^2 = 1$ traversed once in the counterclockwise direction. Evaluate

$$\int_\Gamma \frac{\overbrace{\sin(\pi z^2)}^{f(z)}}{z(z+1)^2}dz.$$



$$\int_\Gamma = \int_{C_\varepsilon(-1)} + \int_{C_\varepsilon(0)}$$

or Partial Fractions:

$$\frac{1}{z(z+1)^2} = \frac{A}{z} + \frac{B}{(z+1)^2} + \frac{C}{(z+1)}$$

$$\int_{C_\varepsilon(0)} \frac{\left[\frac{\sin \pi z^2}{(z+1)^2}\right]}{z-0}\, dz = 2\pi i \, \frac{\sin \pi 0^2}{(0+1)^2} = 0$$

$$\int_{C_\varepsilon(-1)} \frac{\overbrace{\left[\frac{\sin \pi z^2}{z}\right]}^{f(z)}}{(z-(-1))^2}\, dz = \frac{2\pi i}{1!} f'(-1)$$

Answer:

2. (15) (i) Evaluate $\displaystyle\int_C \frac{e^{\sin z} + e^{\bar{z}}}{z^2}\, dz$ where C is the circle $|z| = 1$ traversed once counterclockwise.

$C = \underbrace{e^{it}}_{z(t)}$ $0 \le t \le 2\pi$

$z'(t) = i e^{it}$

$$\int_C \frac{e^{\sin z}}{(z-0)^2}\, dz = \frac{2\pi i}{1!} f'(0) \qquad f(z)$$

$$\int_C \frac{e^{\bar{z}}}{z^2}\, dz = \int_0^{2\pi} \frac{e^{(e^{-it})}}{(e^{it})^2}\, i e^{it}\, dt$$

$$= i\int_0^{2\pi} e^{\cos t} e^{-i \sin t} e^{-it}\, dt$$

$\bar{z} = 1/z$ on C

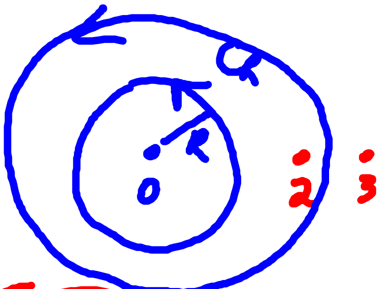
Answer :

(10) (ii) Let L be the line segment from $1 + i$ to $3 + 3i$. Evaluate $\displaystyle\int_L |z|^2\, dz$. Write your answer in $a + ib$ form.

Answer :

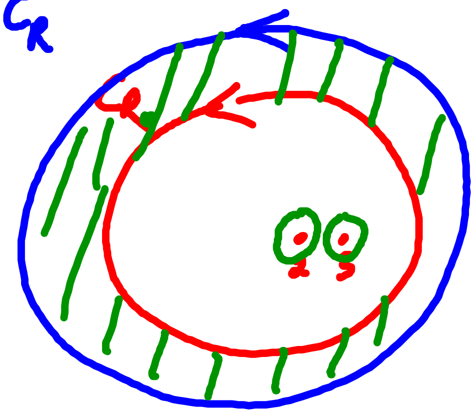
4. (15) For which values of $R > 0$ the integral $\displaystyle\int_C \frac{dz}{(z^2 - 5z + 6)}$, where C is the circle $|z| = R$ traversed once counterclockwise, is equal to zero?

$$(z-2)(z-3) = z^2 - 5z + 6$$



$\int_{C_R} = 0$ by Cauchy's Thm. if $0<R<2$

$$\int_{C_R} \frac{\overbrace{1/(z-3)}^{f(z)}}{z-2}\, dz = 2\pi i\, f(2) \leftarrow \text{not zero if } 2<R<3$$



$$\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$\int_{C_R} = 2\pi i\,(A+B)$$

$$\left|\int_{C_R} \frac{1}{(z-2)(z-3)}\, dz\right| \le \left(\max_{C_R} \frac{1}{|z-2||z-3|}\right)(2\pi R)$$

$$\le \frac{1}{R-2} \cdot \frac{1}{R-3} \cdot 2\pi R$$

$$\to 0 \text{ as } R \to \infty$$

$\int_{C_{R_0}} = \int_{C_R} \to 0$

$\uparrow$ must $= 0$.

Answer :