Lesson 3 on 10.1 Line integrals & work.

MLK Day Monday. No lecture.
HWK 1 due Wed. 1/20, 11:59 pm on Blackboard. Scanned single PDF
WebEx Office Hr. Tues. 8-9 pm

Work:

$\longrightarrow x$

$$F = ma = m\frac{dv}{dt}$$

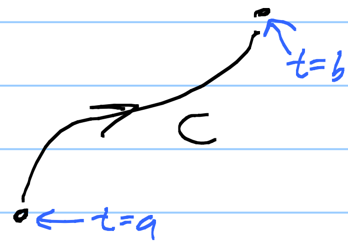
$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} m \underbrace{\frac{dv}{dt}}_{\frac{dv}{dx}\frac{dx}{dt}} dx \qquad x = f(t)$$

$$\frac{dx}{dt} = v$$

$$= m\int_{x_1}^{x_2} v\frac{dv}{dx}\,dx = m\int_{v_1}^{v_2} v\,dv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

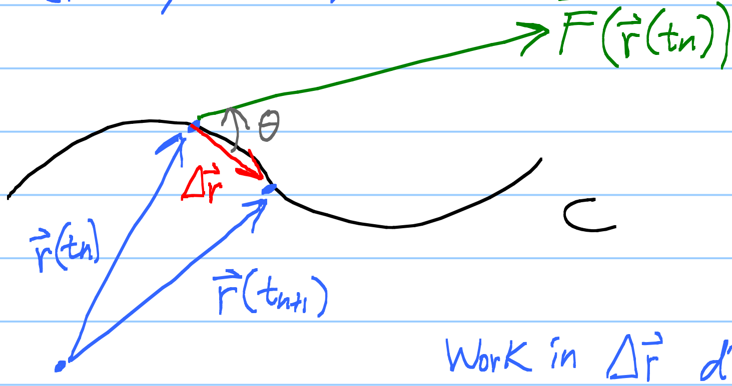
Componentwise:

$$W = \int_C \vec{F}\cdot d\vec{r}$$



$$C:\ \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \qquad a \le t \le b$$

$$W = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$



Work in $\Delta\vec{r}$ dir:

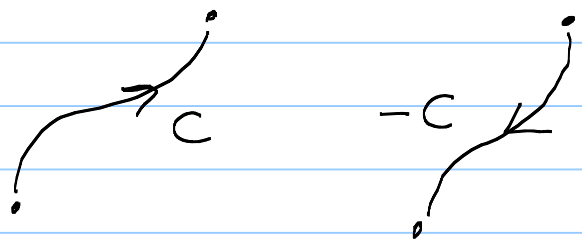
$$= \vec{F}\cdot\Delta\vec{r}$$

$$= \left(\|\vec{F}\|\cos\theta\right)\|\Delta r\|$$

$$\Delta W \cong \vec{F}(\vec{r}(t_n))\cdot\left[\underbrace{\vec{r}(t_{n+1}) - \vec{r}(t_n)}_{\approx\ \vec{r}\,'(t_n)\,\Delta t}\right]$$

$$W = \lim_{\Delta t\to 0}\sum \Delta W = \int \sim\sim$$

Important Fact: W does not depend on parametrization. Only on direction of C.



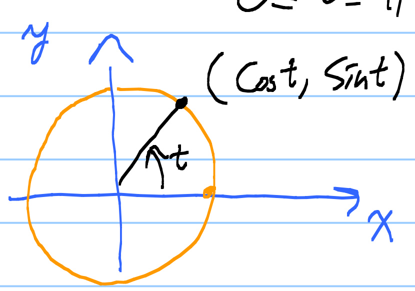
Fact: $$\int_{-C} \vec{F}\cdot d\vec{r} = -\int_{C} \vec{F}\cdot d\vec{r}$$

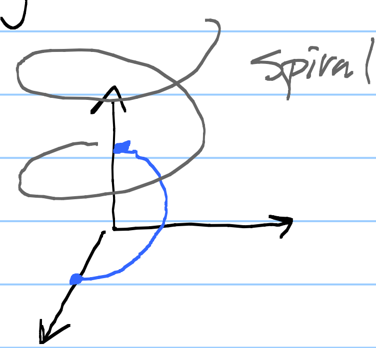
Fact: Div $\vec{V}$ does not depend on choice of cartesian coords. Why: $\text{Div}\,\vec{V} = \dfrac{\text{rate of outflow}}{(\Delta\text{vol})(\Delta t)}$

Also: Curl $\vec{V}$ doesn't depend on coord either.

EX: $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}$

$0 \le t \le \pi$





$$\vec{F} = e^{x}\,\hat{\imath} + xy\,\hat{\jmath} + 1\cdot\hat{k}$$

$$\vec{r}\,'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + 1\cdot\hat{k}$$

$$W = \int_0^{\pi} \vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}\,dt =$$

$$= \int_0^{\pi} \left[e^{\cos t}\,\hat{\imath} + \cos t\,\sin t\,\hat{\jmath} + \hat{k}\right]\cdot\left[-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}\right]dt$$

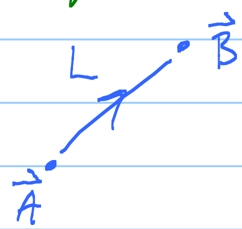
$$= \int_0^{\pi} (-\sin t)\,e^{\overbrace{\cos t}^{u}} + \overset{v=\cos t}{(\cos^2 t)}\,\overset{-dv}{(\sin t)} + 1\;dt$$

$$= \left[e^{\cos t} - \frac{1}{3}\cos^3 t + t \right]_0^{\pi} = e^{-1} - e + \frac{1}{3} - \left(-\frac{1}{3}\right) + \pi$$

$$= e^{-1} - e + \frac{2}{3} + \pi$$

Parametrizing lines, circles, ellipses, graph.

Lines:



$$\vec{r}(t) = \vec{A} + t(\vec{B} - \vec{A})$$

$$0 \le t \le 1$$

EX: Line from $(1,2)$ to $(-1,3)$.

$$\vec{r}(t) = \langle 1,2 \rangle + t\underbrace{\left[\langle -1,3 \rangle - \langle 1,2 \rangle\right]}_{\langle -2,1 \rangle}$$

$$= \underbrace{(1-2t)}_{x}\hat{\imath} + \underbrace{(2+t)}_{y}\hat{\jmath}$$

$$\vec{r}\,'(t) = \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath} = -2\hat{\imath} + \hat{\jmath} = \vec{B} - \vec{A}$$

Prob: Compute $\displaystyle\int_L (x\hat{\imath} + \ln y\,\hat{\jmath}) \cdot d\vec{r}$

$$= \int_0^1 \left[(1-2t)\hat{\imath} + \ln(2+t)\hat{\jmath}\right] \cdot \left[-2\hat{\imath} + \hat{\jmath}\right] dt$$

$$= \int_0^1 4t - 2 + \underbrace{\ln(2+t)}_{u}\ \underbrace{dt}_{dv}$$

$$= \left[2t^2 - 2t + \underline{(2+t)\ln(2+t) - (2+t)}\right]_0^1$$

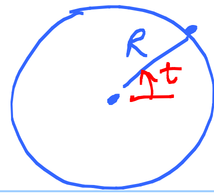
$$= 3\ln 3 - 3 - 2\ln 2 + 2$$

$$= \ln 27 - \ln 4 - 1 = \ln\frac{27}{4} - 1$$

$\ln e = 1$

$$= \ln\frac{27}{4e}$$

Circle:

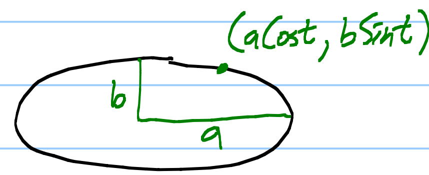


$(R\cos t, R\sin t) = (x,y)$

$$\vec{r}(t) = R\cos t\,\hat{i} + R\sin t\,\hat{j}$$

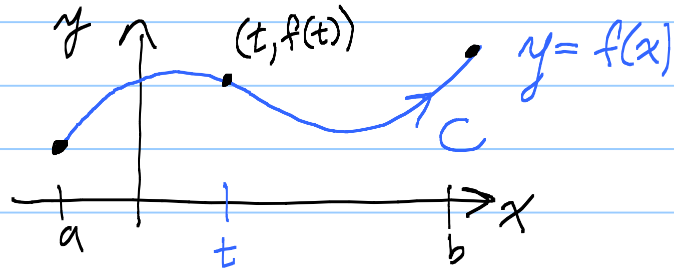
$$x^2 + y^2 = R^2 \underbrace{(\cos^2 t + \sin^2 t)}_{1} = R^2$$

Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



$$\vec{r}(t) = a\cos t\,\hat{i} + b\sin t\,\hat{j}$$

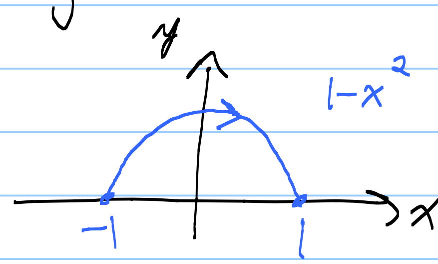
Graph:



Idea: Let $t = x$.

$$\vec{r}(t) = t\,\hat{i} + f(t)\,\hat{j} \qquad a \le t \le b$$

EX: Parametrize parabola



$$\vec{r}(t) = t\,\hat{i} + (1-t^2)\,\hat{j} \qquad -1 \le t \le 1$$

Similar: Sideways pic:

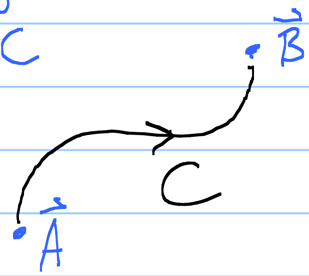
Take $t = y$.

Important case: Parametrize by arc length s.

Notation: $d\vec{r} = d\vec{s}$

Fundamental Theorem of Calculus for line integrals.

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{B}) - f(\vec{A})$$



Why: $\nabla f \cdot \frac{d\vec{r}}{dt} = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \cdot \left[\dot{x}(t), \dot{y}(t), \dot{z}(t)\right]$

$$= \frac{\partial f}{\partial x}\big(x(t), y(t), z(t)\big)\,\dot{x}(t) + \cdots + \frac{\partial f}{\partial z}(\ldots)\,\dot{z}(t)$$

$$= \frac{d}{dt} f\big(x(t), y(t), z(t)\big)$$

$t = a$. Get $f(\vec{A})$

$t = b$. Get $f(\vec{B})$