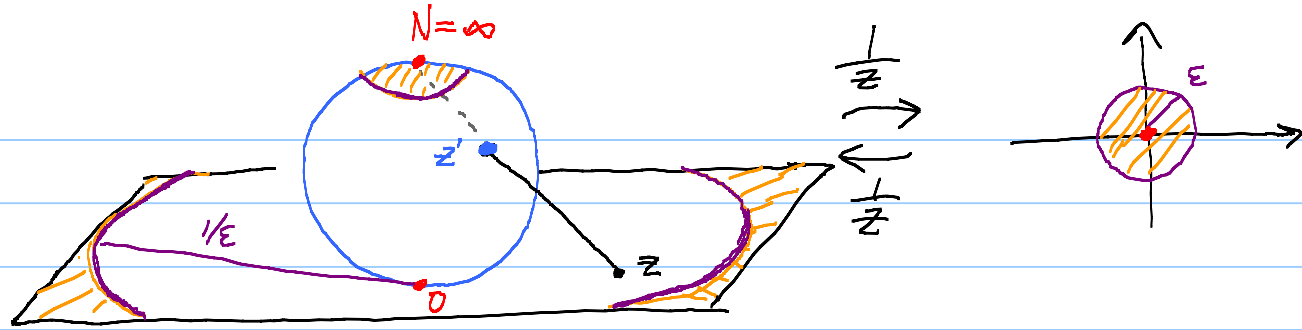
Lesson 32 on 16.3 Residues

HWK 10: Lessons 30, 31, 32 due Wed.



Def^n: If f is analytic on $\underbrace{\{z : |z| > \frac{1}{\varepsilon}\}}_{D_\varepsilon(\infty) - \{\infty\}}$

By convention, the "singularity of f at ∞" has the same type as $f\left(\frac{1}{z}\right)$ at $z=0$.

EX: e^z has an essential sing. at ∞.

$$f\left(\tfrac{1}{z}\right) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

↑ ∞ many neg. terms.

$z=0$ is an ess. sing.

EX: $\frac{z+1}{z-1}$ has a removable sing at ∞.

$$f\left(\tfrac{1}{z}\right) = \frac{\frac{1}{z}+1}{\frac{1}{z}-1} = \frac{1+z}{1-z}$$ ← analytic at $z=0$. $z=0$ is removable.

EX: Polynomial $a_N z^N + \cdots + a_1 z + a_0$ $(N \geq 1)$ has pole of order N at ∞.

Two main ways to recognize types of sing:

1) Get Laurent Expansion.

2) **Fact:** a) If $\lim_{z\to z_0} f = \infty$ ← $\lim_{z\to z_0} |f| = +\infty$

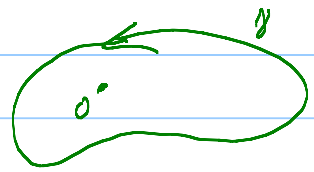
then f has a pole at z_0.

b) If f remains bounded near z_0, or better yet, $\lim_{z\to z_0} f(z)$ exists, then z_0 is a removable sing.

c) If f doesn't do either of the above, then f "shreds" the complex plane near z_0. z_0 is an ess. sing.

EX: $\displaystyle \lim_{z\to\infty} \frac{z+1}{z-1} = \lim_{z\to\infty} \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = \frac{1}{1} = 1$

So ∞ is removable.

EX: 

$\displaystyle \int_\gamma e^{1/z}\,dz =$

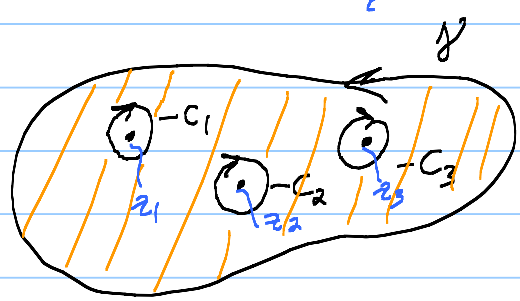
$$\int_\gamma \sum_{n=0}^{\infty} \frac{1}{n!\,z^n}\,dz = 0 + \int_\gamma \frac{1/1!}{z}\,dz + 0 + 0 + 0 + \cdots$$

Residue term

$$= 2\pi i \left(\mathrm{Res}_0\, e^{1/z}\right)$$ ← if 0 inside γ.

Laurent Expansion: $\displaystyle f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

where $\displaystyle a_n = \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{f(w)}{w^{n+1}}\,dw$ ← $\displaystyle a_{-1} = \frac{1}{2\pi i}\int_{C_\varepsilon} f(w)\,dw$



f analytic inside γ except at finitely many points.

$$\left(\int_\gamma + \sum \int_{-C_j}\right) f\,dz = 0$$

$$\int_\gamma f\,dz = \sum \int_{C_j} f\,dz = \sum 2\pi i\, \mathrm{Res}_{z_j} f$$

Residue Theorem: $\displaystyle\int_\gamma f\,dz = 2\pi i \left(\begin{array}{c}\text{Sum of the}\\ \text{Residues of } f\\ \text{inside } \gamma\end{array}\right)$

Finding Residues: Mess around with power series.

Most important case: Suppose f and g are analytic on a disc about z_0, and suppose g has a simple zero at z_0. Then

$$\operatorname{Res}_{z_0}\frac{f}{g} = \frac{f(z_0)}{g'(z_0)} \longleftarrow \underline{\text{simple zero}}: \quad \begin{array}{l} g(z_0)=0 \\ g'(z_0)\neq 0\end{array}$$

Why:

$$g(z) = \underset{\uparrow\; a_0=0}{a_0} + \underset{\uparrow\; a_1=\frac{g'(z_0)}{1!}}{a_1}(z-z_0) + \underset{\uparrow\; a_2=\frac{g''(z_0)}{2!}}{a_2}(z-z_0)^2 + \cdots$$

$$= (z-z_0)\Big[\underbrace{a_1 + a_2(z-z_0) + a_3(z-z_0)^2 + \cdots}_{G(z)}\Big]$$

analytic near z_0. $G(z_0) = a_1 = \dfrac{g'(z_0)}{1!}$

$$\frac{f(z)}{g(z)} = \frac{1}{(z-z_0)}\left[\frac{f(z)}{G(z)}\right]$$

$$= \frac{1}{z-z_0}\Big[A_0 + A_1(z-z_0) + \cdots\Big]$$

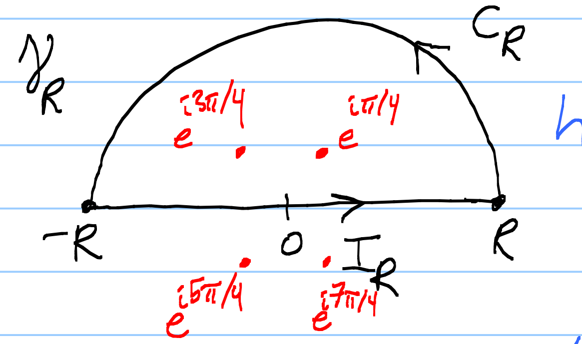
$$= \frac{A_0}{z-z_0} + \underbrace{\Big(A_1 + A_2(z-z_0) + \cdots\Big)}_{\text{power series}}$$

Aha! $\displaystyle \operatorname{Res}_{z_0}\frac{f}{g} = A_0 = \frac{f(z_0)}{G(z_0)} = \frac{f(z_0)}{\left[\frac{g'(z_0)}{1!}\right]} = \frac{f(z_0)}{g'(z_0)}$ ✓

Formula in books: $\displaystyle \operatorname{Res}_{z_0}\frac{f}{g} = \lim_{z\to z_0}(z-z_0)\frac{f(z)}{g(z)}$, when g has a simple zero. Get same result.

Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{t^4+1}\,dt$ ← No antiderivative for $\frac{1}{t^4+1}$!

Trick:



$h(z) = \dfrac{1}{z^4+1}$

$z^4+1=0$ at $z = \underbrace{e^{i\pi/4}, e^{i3\pi/4}}_{\text{inside } \gamma}, \dots, e^{i7\pi/4}$

Residue Thm: $\displaystyle\int_{\gamma_R} h\,dz = 2\pi i\left(\operatorname{Res}_{e^{i\pi/4}} h + \operatorname{Res}_{e^{i3\pi/4}} h\right)$

Note: $h(z) = \dfrac{f(z)}{g(z)} = \dfrac{1}{z^4+1}$ where $f(z) \equiv 1$
$g(z) = z^4+1.$
$g'(z) = 4z^3$

At $z = e^{i\pi/4}$: $g(e^{i\pi/4}) = 0$

$g'(e^{i\pi/4}) = 4(e^{i\pi/4})^3 = \underbrace{4e^{i3\pi/4}}_{\neq 0}$

Simple zero.

So $\operatorname{Res}_{e^{i\pi/4}} h = \dfrac{f(z_0)}{g'(z_0)} = \dfrac{1}{4e^{i3\pi/4}}$

Similarly $\operatorname{Res}_{e^{i3\pi/4}} h = \dfrac{1}{4(e^{i3\pi/4})^3} = \dfrac{1}{4e^{i9\pi/4}} = \dfrac{1}{4e^{i\pi/4}}$

I_R: $z(t) = t$, $-R \le t \le R$.
$z'(t) = 1$

$\displaystyle\int_{I_R} \frac{1}{z^4+1}\,dz = \int_{-R}^{R} \frac{1}{t^4+1}\cdot 1\,dt$ ← I Care about this!

C_R: $z(t) = Re^{it}$, $0 \le t \le \pi$.

$z'(t) = iRe^{it}$

$$\left| \int_{C_R} \frac{1}{z^4+1}\, dz \right| \le \underbrace{\left(\max_{z \in C_R} \left| \frac{1}{z^4+1} \right| \right)}_{\le \frac{1}{R^4-1} \text{ if } R>1} \underbrace{\pi R}_{\text{Length}(C_R)}$$

$$\le \frac{\pi R}{R^4-1} \longrightarrow 0 \text{ as } R \to \infty.$$

<u>Aha!</u> $$\int_{\gamma_R} = \int_{C_R} + \int_{I_R} = 2\pi i \left(\sum \text{Res} \right)$$

$R \to \infty$: $$0 + \int_{-\infty}^{\infty} \frac{1}{t^4+1}\, dt = 2\pi i \left(\frac{1}{4e^{i3\pi/4}} + \frac{1}{4e^{i\pi/4}} \right)$$

$$\int_{-\infty}^{\infty} \frac{1}{t^4+1}\, dt = \frac{\pi i}{2} \left(e^{-i3\pi/4} + e^{-i\pi/4} \right)$$

$$= \pi \sin \pi/4 = \pi \sqrt{2}/2$$

<u>Denom. Estimate:</u> $$|z^4+1| = |z^4 - (-1)|$$

$$\ge \underbrace{\big|\, \underbrace{|z^4|}_{R^4} - |-1| \,\big|}_{|R^4-1| = R^4-1 \text{ if } R>1.}$$

$$\frac{1}{|z^4+1|} \le \frac{1}{R^4-1} \text{ for } z \in C_R \text{ if } R>1.$$