Lesson 33 on 16.4 Residues (part 2)

HWK 10: Lessons 30,31,32 due Wed

This week: Office Hours M,T 2-3pm. (No office hour on Wed.)
WebEx Office Hour Tues. 8-9pm

$$\operatorname{Res}_{z_0} \frac{f}{g} = \frac{f(z_0)}{g'(z_0)}$$ ← true when g has a simple zero at z_0 and f is analytic near z_0.

EX: $\operatorname{Res}_{\frac{\pi}{2}} \operatorname{Tan} z$ $\quad \operatorname{Tan} z = \dfrac{\sin z}{\cos z}$ ← f, ← g

$g\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$ ← zero $\qquad g'(z) = -\sin z$

$g'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$ ← not zero.

$\frac{\pi}{2}$ is order 1 zero. (simple).

$$\operatorname{Res}_{\frac{\pi}{2}} \operatorname{Tan} z = \frac{f(z_0)}{g'(z_0)} = \frac{\sin\frac{\pi}{2}}{\left(-\sin\frac{\pi}{2}\right)} = \frac{1}{-1} = -1$$

EX: $\dfrac{f}{g}$ where g has a double zero at z_0.

double zero $\begin{cases} g(z_0) = 0 \\ g'(z_0) = 0 \\ g''(z_0) \neq 0 \end{cases}$

$$g(z) = \underbrace{a_0 + a_1(z-z_0)}_{0} + a_2(z-z_0)^2 + \cdots$$

$$= (z-z_0)^2 \Big[\underbrace{a_2 + a_3(z-z_0) + \cdots}_{G(z)}\Big]$$

G is analytic near z_0 and $\boxed{G(z_0) = a_2 = \frac{1}{2!} g''(z_0)}$

not zero!

$\boxed{G'(z_0) = 1!\, a_3 = \frac{1}{3!} g'''(z_0)}$

Near z_0: $$\frac{f(z)}{g(z)} = \frac{1}{(z-z_0)^2}\Bigg[\underbrace{\frac{f(z)}{G(z)}}_{H(z)}\Bigg]$$

$H(z)$ ← analytic near z_0

$$= \frac{1}{(z-z_0)^2}\Big[A_0 + A_1(z-z_0) + A_2(z-z_0)^2 + \cdots\Big]$$

pole of order 2 → $$= \frac{A_0}{(z-z_0)^2} + \frac{A_1}{(z-z_0)} + \text{power series}$$

Aha! $\operatorname{Res}_{z_0} \frac{f}{g} = A_1 = \frac{H'(z_0)}{1!} = H'(z_0) =$

$$\boxed{\begin{aligned} G(z_0) &= \frac{g''(z_0)}{2!} \\ G'(z_0) &= \frac{g'''(z_0)}{3!} \end{aligned}}$$

$$= \left[\frac{f}{G}\right]'\Bigg|_{z_0} = \frac{f'(z_0)\,G(z_0) - f(z_0)\,G'(z_0)}{G(z_0)^2}$$

$$= \frac{f'(z_0)\,g''(z_0)/2 - f(z_0)\,g'''(z_0)/6}{\left[g''(z_0)/2\right]^2}$$

EX: $\operatorname{Res}_0 \csc z \cot z$

$$\csc z \cot z = \frac{1}{\sin z}\,\frac{\cos z}{\sin z} = \frac{\cos z}{\sin^2 z}$$

$$= \frac{1 - \frac{z^2}{2!} + \cdots}{\left(z - \frac{z^3}{3!} + \cdots\right)^2} = \frac{1}{z^2} \cdot \underbrace{\left[\frac{1 - \frac{z^2}{2!}}{\left(1 - \frac{z^2}{3!} + \cdots\right)^2}\right]}_{A_0 + A_1 z + \cdots}$$

Need A_1

Alternative Formula:

$$\frac{f}{g} = \frac{1}{(z-z_0)^2} \cdot \frac{f}{G} = \frac{A_0}{(z-z_0)^2} + \frac{A_1}{(z-z_0)} + \cdots$$

Big Idea: Multiply by $(z-z_0)^2$:

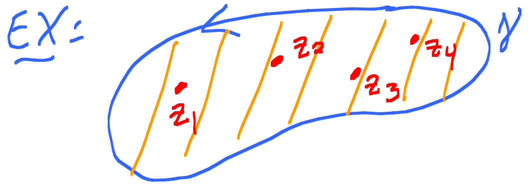
$$(z-z_0)^2 \frac{f(z)}{g(z)} = \underbrace{A_0 + A_1(z-z_0) + A_2(z-z_0)^2 + \cdots}_{H(z),\ \text{Regular power series}}$$

$$A_1 = \frac{H'(z_0)}{1!} = \lim_{z \to z_0} \frac{H'(z)}{1!}$$

Aha! $\operatorname{Res}_{z_0} \frac{f}{g} = \lim_{z \to z_0} \frac{1}{1!} \frac{d}{dz}\left[(z-z_0)^2 \frac{f(z)}{g(z)}\right]$

Get same answer. Use limits like L'H rule. More work.

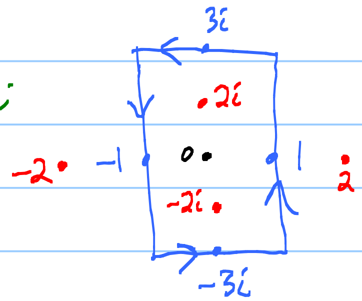
EX:



f analytic inside and on γ <u>except</u> at finitely many points inside γ.

Residue Thm: $\displaystyle\int_\gamma f\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}_{z_j} f$

EX:



$$\int_\gamma \frac{z e^{\pi z}}{z^4-16}\,dz$$

$$= 2\pi i \sum \left(\operatorname{Res}_{\pm 2i} \frac{z e^{\pi z}}{z^4-16} \right)$$

$$z^4 - 16 = 0$$

$$z^4 = 16$$

$$(re^{i\theta})^4 = 16$$

$$r^4 e^{i4\theta} = 16 e^{i0} \leftarrow r^4 = 16 \quad \boxed{r=2}$$

$$e^{i4\theta} = e^{i0} \qquad 4\theta = 0 + 2n\pi,\ n = 0, \pm 1, \pm 2, \dots$$

$$\theta = n\frac{\pi}{2},\ n \in \mathbb{Z}.$$

$$z = \underset{\substack{\uparrow \\ \text{outside box}}}{\pm 2},\ \underset{\substack{\uparrow \\ \text{inside}}}{\pm 2i}$$

$$\frac{z e^{\pi z}}{z^4-16} \quad \begin{matrix} \leftarrow f \\ \leftarrow g \end{matrix}$$

$$g(2i) = 0$$

$$g'(2i) = 4z^3 \Big|_{z=2i} = 4(2i)^3$$

g simple zero at $2i$

$$\operatorname{Res}_{2i} \frac{z e^{\pi z}}{z^4-16} = \frac{(2i)e^{\pi 2i}}{4(2i)^3} = -\frac{1}{16}$$

$$\operatorname{Res}_{-2i} = \frac{(-2i)e^{\pi(-2i)}}{4(-2i)^3} = -\frac{1}{16}$$

$$\int_\gamma = 2\pi i \left(-\frac{1}{16} - \frac{1}{16} \right)$$

EX: $\displaystyle\int_\gamma \left(\frac{z e^{\pi z}}{z^4-16} + e^{\pi/z}\right) dz$

$\uparrow$ Done

$\displaystyle\int_\gamma e^{\pi/z}\, dz$

$e^{(\pi/z)} = 1 + (\pi/z) + \dfrac{(\pi/z)^2}{2!} + \cdots$

$= 2\pi i \left[\pi\right]$

$\uparrow$ $\mathrm{Res}_0\, e^{\pi/z}$

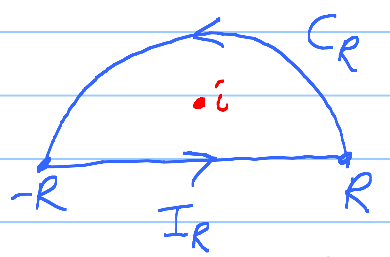
$= 1 + \dfrac{\pi}{z} +$ higher order stuff

EX: $I = \displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx$

Note: $\displaystyle\int_{-\infty}^{\infty} \underbrace{\frac{\sin x}{x^2+1}}_{\text{odd}} dx = 0$

$I = \displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\, dx$

$f(z) = \dfrac{e^{iz}}{z^2+1}$



$\displaystyle\int_{I_R} f(z)\, dz = \int_{-R}^{R} \frac{e^{it}}{t^2+1}\, dt \leftarrow$ Want! as $R \to \infty$.

On C_R: $e^{iz} = e^{i(x+iy)} = e^{ix-y} = e^{-y} e^{ix}$

Aha! $|e^{iz}| = e^{-y} \leq 1$

$|e^{ix}| = 1$

$\uparrow$ because $y \geq 0$ on C_R

Denom. Estimate: $|z^2+1| = |z^2-(-1)| \geq \big|\, |z^2| - |-1| \,\big|$

$\uparrow$ $|R^2-1| = R^2-1$

$\uparrow$ if $R>1$.

On C_R, if $R>1$, $\dfrac{1}{|z^2+1|} \leq \dfrac{1}{R^2-1}$.

So $$\left| \int_{C_R} \frac{e^{iz}}{z^2+1}\, dz \right| \leq \frac{1}{R^2-1} \cdot (\pi R)$$

$$\to 0 \text{ as } R \to \infty.$$

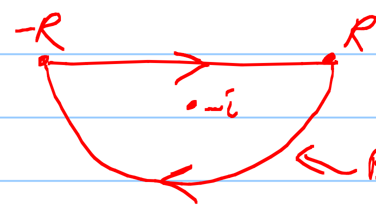
$$\int_{C_R} + \int_{I_R} = 2\pi i \operatorname{Res}_i \frac{e^{iz}}{z^2+1}$$

$R \to \infty$ $\downarrow$ $\downarrow$

$$0 + I = 2\pi i \frac{e^{i(i)}}{2(i)} = \frac{\pi}{e}$$

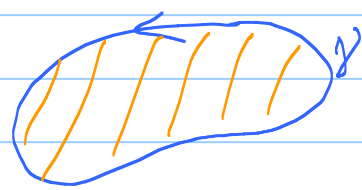
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx = \frac{\pi}{e} \;!$$

Bad choice:



Bad! e^{iz} not bounded when $y<0$.

Argument Principle:



f analytic inside γ, non-vanishing on γ

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f}\, dz = \left(\begin{array}{c} \#\text{ of zeroes of} \\ f \text{ inside } \gamma \\ \text{counting mult.} \end{array} \right)$$

Why: $f(z) = (z-z_0)^N F(z)$ $\quad F(z_0) \neq 0$.

$$\frac{f'(z)}{f(z)} = \frac{N(z-z_0)^{N-1} F(z) + (z-z_0)^N F'(z)}{(z-z_0)^N F(z)}$$

$$= \frac{N}{(z-z_0)} + \frac{F'(z)}{F(z)}$$ ← nice near z_0

$$\operatorname{Res}_{z_0} \frac{f'}{f} = \text{Order of zero of } f \text{ at } z_0!$$