Lesson 34 on 16.4 More Residue integrals

(No Office Hour today)

HWK 10: Lessons 30, 31, 32 due tonight 11:59 pm

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad e^{-i\theta} = \frac{1}{e^{i\theta}}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

EX: $\displaystyle\int_0^{2\pi} \cos^6\theta \, d\theta = I$

Hmmm. $z(\theta) = e^{i\theta}$ parametrizes unit circle C.

$z'(\theta) = ie^{i\theta}$

$dz = z'(\theta)d\theta = ie^{i\theta}d\theta$

$$I = \int_0^{2\pi} \left[\frac{\overset{z}{e^{i\theta}} + \frac{1}{\overset{z}{e^{i\theta}}}}{2}\right]^6 \underbrace{\frac{ie^{i\theta}}{\underset{z}{ie^{i\theta}}} d\theta}_{dz}$$

$$= \int_C \left[\frac{z+\frac{1}{z}}{2}\right]^6 \frac{1}{iz}\,dz \longleftarrow \text{Use Residue Thm!}$$

$$= \frac{1}{i2^6}\int_C \left(\sum_{k=0}^{6}\binom{6}{k}\underbrace{z^{-k}z^{6-k}}_{\text{constant}}\right)\frac{1}{z}\,dz$$

constant term: $2k=6$, $k=3$

$$= \frac{1}{i2^6}\cdot 2\pi i \cdot \underbrace{\binom{6}{3}}_{20} = \frac{1}{2^6}2\pi(2\cdot 2\cdot 5) = \frac{5\pi}{8}$$

$$\text{or} = \frac{1}{i2^6}\int_C \left(z^6 + 6z^4 + 15z^2 + 20 + 15\frac{1}{z^2} + 6\frac{1}{z^4} + \frac{1}{z^6}\right)\frac{1}{z}\,dz$$

Res_0

Fact: $R(z,w)$ rational.

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta = \int_C R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{1}{iz}\,dz$$

$$= 2\pi i \sum(\text{Residues inside } C).$$

EX: $\displaystyle\int_0^{2\pi} \frac{1}{\sqrt{2}-\cos\theta}\,d\theta$ (p. 726)

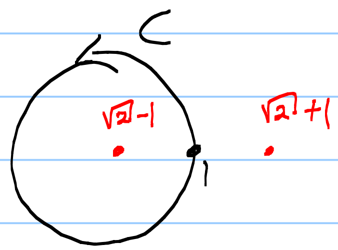
$$\int_C \frac{1}{\left(\sqrt{2} - \frac{z+\frac{1}{z}}{2}\right)} \cdot \frac{1}{iz}\,dz$$

$$= \frac{-2}{i}\int_C \frac{1}{z^2 - 2\sqrt{2}z + 1}\,dz$$

$f(z) = 1$

$g(z) = z^2 - 2\sqrt{2}z + 1$

Quad. Form: Roots denom: $\dfrac{2\sqrt{2} \pm \sqrt{8-4}}{2} = \sqrt{2} \pm 1$



$$I = \left(2\pi i \operatorname{Res}_{\sqrt{2}-1} \frac{f}{g}\right)\left(\frac{-2}{i}\right)$$

$$= \left(\frac{-2}{i}\right) \cdot 2\pi i \frac{f(\sqrt{2}-1)}{g'(\sqrt{2}-1)} = 2\pi i \frac{1}{(-2)}\left(\frac{-2}{i}\right) = \underline{\underline{2\pi}}$$

<u>Claim:</u> $\sqrt{2}-1$ is a <u>simple</u> zero of g.

Why: $g'(z) = 2z - 2\sqrt{2}$

$g'(\sqrt{2}-1) = 2(\sqrt{2}-1) - 2\sqrt{2} = -2 \neq 0$

<u>Book:</u> Or factor denom: $\dfrac{1}{(z-(\sqrt{2}-1))(z-(\sqrt{2}+1))}$

$$\frac{1}{z-(\sqrt{2}-1)}\left[\underbrace{\frac{1}{z-(\sqrt{2}+1)}}\right]$$

$H(z) = A_0 + A_1(z-(\sqrt{2}-1)) + \cdots$

$$= \frac{A_0}{z-(\sqrt{2}-1)} + A_1 + \cdots$$

Aha! $\operatorname{Res} = A_0 = H(\sqrt{2}-1)$

Denom. Estimate: $\left|\dfrac{1}{z^4+1}\right| \leq \dfrac{1}{R^4-1}$ $\quad |z| = R > 1.$

<u>Basic Estimate for Polynomials:</u>

$$p(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$$

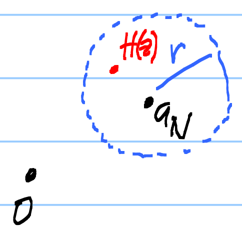
where $a_N \neq 0$ and $N \geq 1$.

There are constants $0 < A < B$ and a radius $R > 0$ such that

$$A|z|^N \leq |p(z)| \leq B|z|^N \quad \text{if } |z| \geq R.$$

Why: $p(z) = z^N \Big[\underbrace{a_N + \frac{a_{N-1}}{z} + \cdots + \frac{a_0}{z^N}}_{H(z)} \Big]$

Notice that $\lim\limits_{z \to \infty} H(z) = a_N$



Take $r = \frac{|a_N|}{2}$

There is an $R > 0$ such that $\left| H(z) - a_N \right| < \frac{|a_N|}{2}$ if $|z| > R$.

Get basic estimate using $A = \frac{|a_N|}{2}$, $B = \frac{3|a_N|}{2}$.

$$A < |H(z)| < B \quad \text{if } |z| > R$$

$$A|z|^N < |\underbrace{z^N H(z)}_{p(z)}| < B|z|^N \quad , |z| > R$$

Denom. Estimate.

Numerator estimate

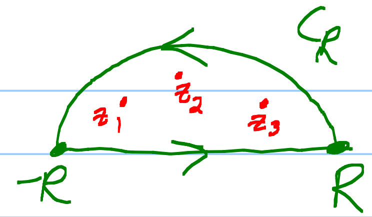
Thm: $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum \mathrm{Res}_{z_j} \frac{P}{Q}$

where Σ is over zeroes of Q in the upper half plane.

Need: 1) No zeroes of Q on $\mathbb{R}$.

2) $\deg Q \geq \deg P + 2$

Why:



$$\left| \int_{C_R} \frac{P}{Q}\, dz \right| \leq \left(\max_{|z|=R} \frac{|P(z)|}{|Q(z)|} \right) \pi R$$

$$\leq \frac{B \cdot R^N}{A \cdot R^M} \cdot \pi R$$

when $R > \max(R_P, R_Q)$

$$\leq C \frac{1}{R^{M-N-1}} \qquad C = \frac{\pi B}{A}$$

$\to 0$ as $R \to \infty$.

$$M - N - 1 = \underbrace{\deg Q - \deg P}_{\geq 2} - 1 \geq 2 - 1 = \underline{1}$$

$$\int_{I_R} + \int_{C_R} = 2\pi i \sum_{\text{inside}} \operatorname{Res} \frac{P}{Q}$$

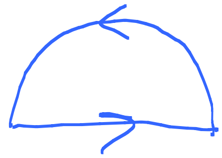
$$\downarrow \qquad \downarrow$$

$$(\text{Want}) + 0 = \checkmark$$

Similar Fact: $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos sx \, dx \quad (s > 0)$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin sx \, dx \quad (s > 0)$$

Kill both with $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{isx} \, dx$. $\left(\begin{array}{l} \text{Top} = \text{Re} \\ \text{Bottom} = \text{Im} \end{array} \right)$



$$f(z) = \frac{P(z)}{Q(z)} e^{isz}$$

$$\left| e^{is(x+iy)} \right| = \underbrace{\left| e^{isx} \right|}_{1} e^{-sy} = e^{-sy}$$

< 1 in UHP

Basic Poly Estimate shows

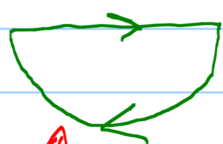
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{isx}\, dx = 2\pi i \sum \text{Res} \frac{P(z)}{Q(z)} e^{isz}$$

zeroes of Q in UHP.

Need: 1) No zeroes of Q on $\mathbb{R}$

2) $\deg Q \geq \deg P + 2$

Remark: For $s < 0$, $\left| e^{isz} \right| \leq 1$ in Lower H.P.



clockwise!

$$\underbrace{\int_{I_R}}_{\text{(want)}} + \underbrace{\int_{C_R}}_{0} = -2\pi i \sum \text{Res} \frac{P}{Q} e^{isz}$$

zeroes of Q in LHP

HWK Prob. $\underbrace{e^{1/(z-1)}}_{\text{ess. sing at } z=1} \dfrac{1}{(e^z - 1)} \leftarrow$ poles where $e^z - 1 = 0$

Take seq $z_n \to 1$. $e^{1/(z-1)} = \begin{cases} \text{Big} \\ \text{Small} \end{cases}$ times $\frac{1}{(e^1 - 1)}$

See no limit as $z \to 1$. No pole.