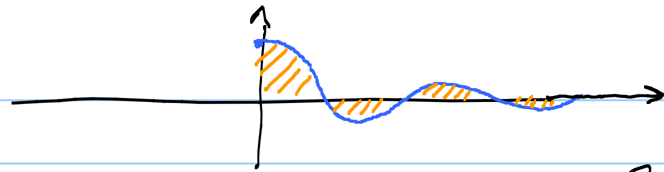
Lesson 35 on 16.4 Real integrals via complex analysis

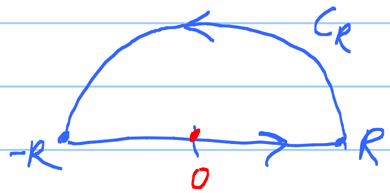
HWK 11: Lessons 33, 34, 35

EX: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin t}{t}\,dt$



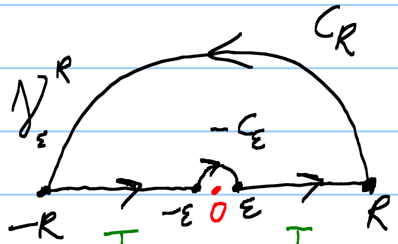
Alternating series test shows $\int$ converges.
(Not absolutely convergent.)

Bad idea: Take $g(z) = \frac{\sin z}{z}$



Ouch! $\sin z$ is very unbounded in UHP.

Good idea: Take $f(z) = \frac{e^{iz}}{z}$ ← analytic inside and on γ_ε^R



$$\left(\int_{I_L} + \int_{I_R}\right) f(z)\,dz = \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) \underbrace{\overset{\text{odd}}{\frac{\cos t}{t}} + i\overset{\text{even}}{\frac{\sin t}{t}}}_{\frac{e^{it}}{t}}\,dt$$

$$= 2i\int_{\varepsilon}^{R} \frac{\sin t}{t}\,dt \leftarrow \text{Want!}$$

Near 0: $\displaystyle\frac{e^{iz}}{z} = \frac{\left[1 + (iz) + \frac{(iz)^2}{2!} + \cdots\right]}{z}$

$$= \frac{1}{z} + \underbrace{i - \frac{1}{2!}z + \cdots}_{\substack{\text{analytic,}\\ \text{entire!}\\ \text{Call it } H(z).}}$$

$$= \frac{1}{z} + H(z)$$

Let $\displaystyle M = \max_{|z|\le 1} |H|.$

$$\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz = -\int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz$$

$$= \underbrace{-\int_{C_\varepsilon} \frac{1}{z}\,dz}\; - \underbrace{\int_{C_\varepsilon} H(z)\,dz}$$

$$= -\int_0^{\pi} \frac{1}{\varepsilon e^{it}} \underbrace{i\varepsilon e^{it}\,dt}_{dz} \qquad \left|\int_{C_\varepsilon} H\,dz\right|$$

$$\leq M(\pi\varepsilon)$$

$$= -\pi i \qquad \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

$$\longrightarrow -\pi i \text{ as } \varepsilon \to 0.$$

On C_R: $\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| \leq \frac{1}{R}\cdot(\pi R) = \pi$

blast!

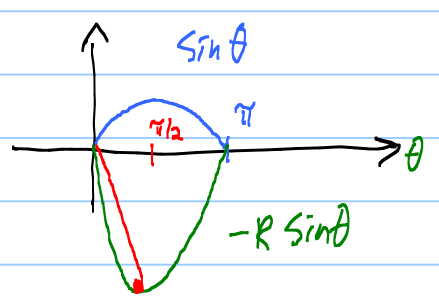
Basic estimate fails us.

$$\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| = \left|\int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}} \underbrace{iRe^{it}\,dt}_{dz}\right|$$

$$= \left|\int_0^{\pi} e^{i(R\cos t + iR\sin t)}\,dt\right|$$

$$\underbrace{\leq \int_0^{\pi} \left|\underbrace{e^{iR\cos t}}_{\substack{\text{modulus}\\ =1}}\, e^{-R\sin t}\right| dt}$$

$$= \int_0^{\pi} e^{-R\sin t}\,dt$$



$$= 2\int_0^{\pi/2} e^{-R\sin t}\,dt$$

$$-R\sin\theta \leq -R\frac{2}{\pi}\theta$$

So $$\int_0^{\pi} e^{-R\sin t}\,dt \le \underbrace{2\int_0^{\pi/2} e^{-R\frac{2}{\pi}t}\,dt}$$

$$2\left[-\frac{\pi}{2R}e^{-R\frac{2}{\pi}t}\right]_0^{\pi/2}$$

$$\frac{\pi}{R}\left(1-e^{-R}\right)$$

$$\le \frac{\pi}{R} \to 0 \text{ as } R\to\infty.$$

Cauchy Theorem!

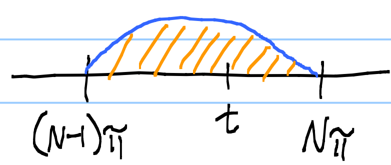
$$\left(\int_{C_R} + \underbrace{\left(\int_{I_L}+\int_{I_R}\right)} + \int_{-C_\varepsilon}\right) f(z)\,dz = 0$$

$$\downarrow R\to\infty \qquad \downarrow \qquad \downarrow \varepsilon\to 0$$

$$0 + 2i\int_0^{\infty}\frac{\sin t}{t}\,dt \quad -\pi i = 0$$

$$\int_0^{\infty}\frac{\sin t}{t}\,dt = \frac{\pi}{2} \quad \text{and} \quad \int_{-\infty}^{\infty}\frac{\sin t}{t}\,dt = \pi$$

Remark:

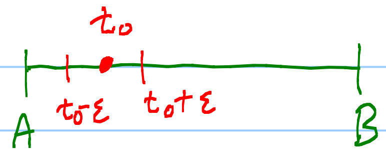


$$\frac{1}{t} \ge \frac{1}{N\pi}$$

$$\int_{(N-1)\pi}^{N\pi}\frac{\sin t}{t}\,dt \ge \frac{1}{N\pi}\underbrace{\int_{(N-1)\pi}^{N\pi}\sin t\,dt}_{2} \ge \frac{2}{N\pi}$$

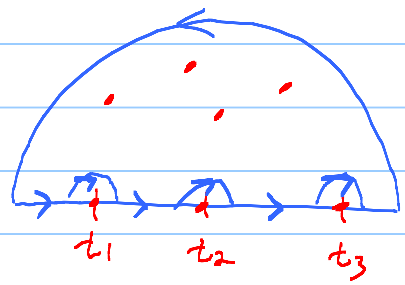
Ouch! $$\int_0^{\infty}\left|\frac{\sin t}{t}\right|dt > \sum_{N=1}^{\infty}\frac{2}{N\pi} = \infty.$$

Standard tool: Cauchy Principal Value



$$PV\int_A^B f(t)\,dt = \lim_{\varepsilon\to 0}\left(\int_A^{t_0-\varepsilon} + \int_{t_0+\varepsilon}^{B}\right) f\,dt$$

Fact:



$f(z)$

$$PV\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \underbrace{\sum \operatorname{Res} f}_{\text{singularities in UHP}} + \underbrace{\pi i \sum \operatorname{Res}_{t_j} f}_{\text{Half the usual Res. Thm!}}$$

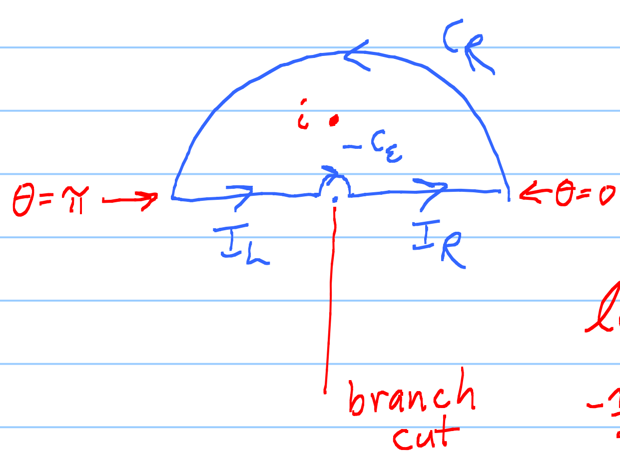
along $\mathbb{R}$

Need: Simple poles on $\mathbb{R}$

Finitely many poles in UHP

$f(z) = \dfrac{P(z)}{Q(z)}$ rational. $\deg Q \geq \deg P + 2$.

EX: $\displaystyle\int_0^\infty \frac{x^\alpha}{x^2+1}\,dx$, $0 < \alpha < 1$.



$x^\alpha = e^{\alpha \ln x}$

$z^\alpha = e^{\alpha \log z}$

$\log z = \ln|z| + i\theta$

$-\frac{\pi}{2} < \theta = \arg z < \frac{3\pi}{2}$

$$f(z)=\frac{e^{\alpha \log z}}{z^2+1}$$

$$\left|e^{\alpha \log z}\right| = \left|e^{\alpha \ln|z| + i\alpha\theta}\right| = e^{\ln|z|^{\alpha}}\underbrace{\left|e^{i\alpha\theta}\right|}_{1} = |z|^{\alpha}$$

$$\boxed{\left|z^{\alpha}\right| = |z|^{\alpha}}$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^{\alpha}}{R^2-1}(\pi R) \longrightarrow 0$$

as $R\to\infty$.

$(\alpha<1)$.

<u>On C_ε:</u> Denom estimate $|z^2+1| = |z^2-(-1)|$

$|z|=\varepsilon$ $\qquad \ge \underbrace{\left||z|^2-1\right|}_{\left|\varepsilon^2-1\right| = 1-\varepsilon^2}$

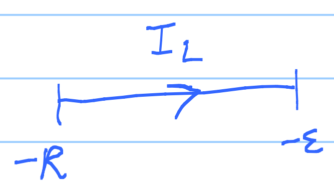
$$\left|\int_{-C_\varepsilon} \frac{e^{\alpha \log z}}{z^2+1}\,dz\right| \le \frac{\varepsilon^{\alpha}}{1-\varepsilon^2}\cdot(\pi\varepsilon)$$

$$\longrightarrow 0 \text{ as } \varepsilon\to 0.$$

<u>Finally:</u>

$$\Big(\underbrace{\int_{C_R}}_{\downarrow\, 0} + \underbrace{\int_{I_L}}_{\downarrow\, ?} + \underbrace{\int_{I_R}}_{\downarrow\, \text{want}} + \underbrace{\int_{-C_\varepsilon}}_{\downarrow\, 0}\Big) f\,dz = 2\pi i \operatorname{Res}_i \frac{e^{\alpha \log z}}{z^2+1}$$

$$= 2\pi i\,\frac{e^{\alpha \log i}}{2i} = \pi e^{\alpha i \pi/2}$$



$$\frac{e^{\alpha \log t}}{t^2+1} = e^{\alpha(\ln|t| + i\pi)}\Big/ t^2+1$$

$$= e^{\alpha \ln|t|}\, e^{\alpha i\pi} \Big/ t^2+1$$

$$= |t|^{\alpha}\, e^{i\alpha\pi}\Big/ t^2+1$$

So $\int_{I_L} f(z)\,dz = e^{i\alpha\pi} \int_{\varepsilon}^{R} \frac{t^\alpha}{t^2+1}\,dt$

Put it all together:

$$\left(1+e^{i\alpha\pi}\right)\int_0^\infty \frac{t^\alpha}{t^2+1}\,dt = \pi e^{i\alpha\pi/2}$$

$$I = \pi \frac{e^{i\alpha\pi/2}}{1+e^{i\alpha\pi}} \cdot \frac{e^{-i\alpha\pi/2}}{e^{-i\alpha\pi/2}}$$

$$= \frac{\pi}{2\cos \alpha\pi/2}$$