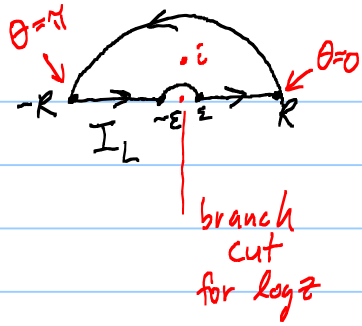
Lesson 36 on 17.1 Conformal Mapping

HWK 11: Lessons 33, 34, 35 due Wed.

WebEx Office Hour Wed., 8–9 pm



$f(z) = \dfrac{e^{\alpha \log z}}{z^2+1}$

$I_L : z(t) = t, \quad -R \le t \le -\varepsilon$

$$\boxed{\text{Way I prefer: } -I_L : z(t) = -t, \quad \varepsilon \le t \le R}$$

$z'(t) = -1$

$$\int_{I_L} f(z)\,dz = -\int_{-I_L} f(z)\,dz = -\int_{\varepsilon}^{R} \frac{e^{\alpha(\ln|-t| + i\pi)}}{(-t)^2+1} \underbrace{(-1)\,dt}_{z'(t)dt}$$

$$= e^{i\alpha\pi} \underbrace{\int_{\varepsilon}^{R} \frac{e^{\alpha \ln t}}{t^2+1}\,dt}_{\text{Want}} \qquad e^{\alpha \ln t} = t^\alpha$$

Chain Rule #2:

$f(z)$ analytic. $z(t) = x(t) + i y(t)$. $\qquad f = u + iv$

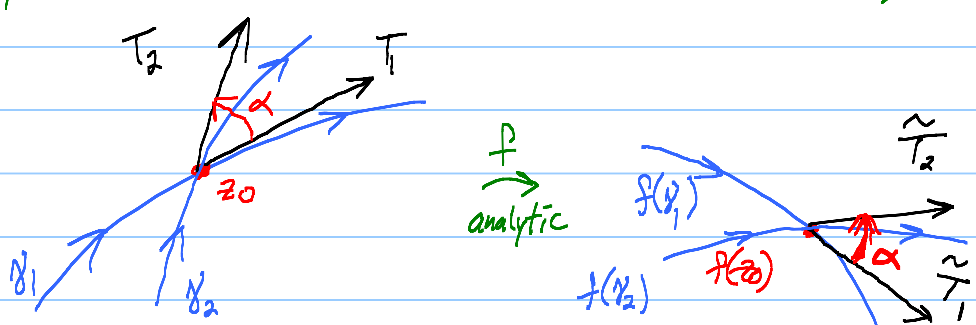
$$\boxed{\frac{d}{dt} f(z(t)) = f'(z(t))\, z'(t)}$$

Why: $\dfrac{d}{dt}\Big[u(x(t), y(t)) + i\, v(x(t), y(t))\Big]$

$$= (u_x \dot{x} + \underset{= -v_x}{u_y} \dot{y}) + i\,(v_x \dot{x} + \underset{= u_x}{v_y} \dot{y})$$

$$= \underbrace{(u_x + i v_x)}_{f'} (\dot{x} + i \dot{y}) \quad \checkmark$$

Analytic fcns are conformal: (Where $f'(z) \neq 0$)



$\gamma_j : z_j(t), \quad -1 \le t \le 1$

$z(0) = z_0$

$f(\gamma_j) : f(z_j(t)), \quad -1 \le t \le 1$

$$T_j = z_j'(0) \qquad \tilde{T}_j = \frac{d}{dt} f(z_j(t)) \Big|_{t=0}$$

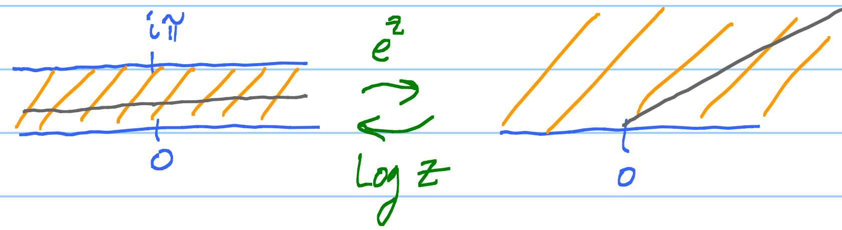
$$= f'(z_j(t))\, z_j'(t) \Big|_{t=0}$$

$$\tilde{T}_j = \underbrace{f'(z_0)}_{Re^{i\theta}} T_j$$

Aha! α is preserved!
And so is the sense.

Stretch by R — Rotate by θ

EX:



Linear Fractional Transformations (LFTs)

$$L(z) = \frac{az+b}{cz+d} \qquad (ad-bc \neq 0)$$

Fact: LFTs are composed of the basic maps

1) $z \mapsto rz \quad (r \in \mathbb{R}^+)$ Stretch
2) $z \mapsto e^{i\theta} z$ Rotation
3) $z \mapsto z+b \quad (b \in \mathbb{C})$ Translation
4) $z \mapsto 1/z$ Inversion

Why: $c=0$ case: $L(z) = Az + B$

$A = Re^{i\theta}$ — Translate

$$\left(\frac{az+b}{d} \quad A = \frac{a}{d},\ B = \frac{b}{d} \right.$$

1. Stretch by R
2. Rotate by θ
3. Translate by B

$re^{i\alpha}$

$c \neq 0$ case: $\dfrac{az+b}{cz+d} = \dfrac{a}{c} + \dfrac{\left(b - \frac{ad}{c}\right)}{cz+d}$ ← long division

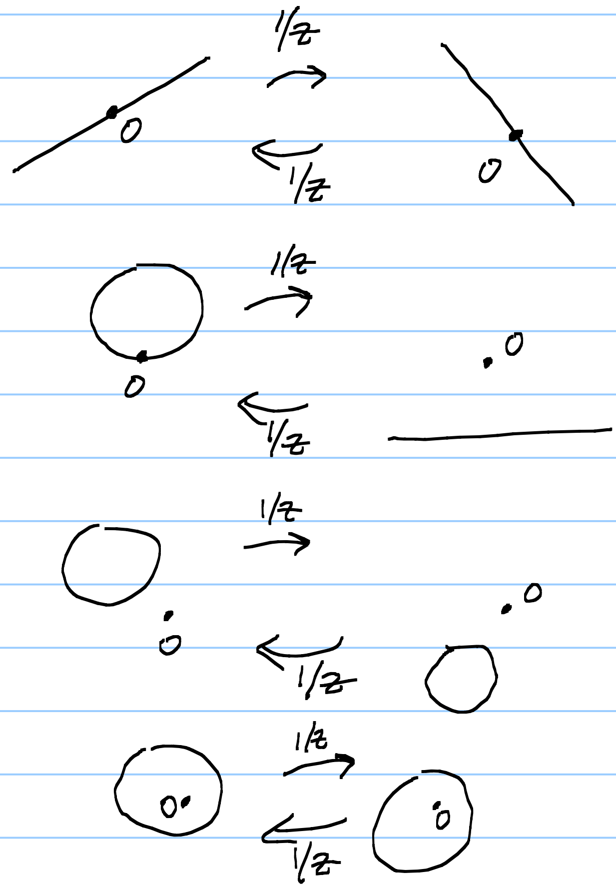
$\uparrow$ $Re^{i\theta}$

7 steps! using 1-4.

Observe: 1-3 take $\{\text{lines}\} \to \{\text{lines}\}$, $\{\text{circles}\} \to \{\text{circles}\}$ } easy

analytic geometry → 4: $\{\text{lines, circles}\} \to \{\text{lines, circle}\}$

Picture demo:



Consequence: LFTs preserve $\{\text{lines, circles}\}$

Def^n: "Extended complex plane"

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Fact: LFTs map $\hat{\mathbb{C}}$ one-to-one onto itself.

Why: $c \neq 0$ $\quad L(z) = \dfrac{az+b}{cz+d} \cdot \dfrac{\left(\frac{1}{z}\right)}{\left(\frac{1}{z}\right)}$ $\quad \lim_{z \to \infty} L(z) = \dfrac{a}{c}$

Say $L(\infty) = \dfrac{a}{c}$

Simple pole at $z = -\dfrac{d}{c}$. Know $\lim_{z \to -\frac{d}{c}} L(z) = \infty$.

Say $L\left(-\dfrac{d}{c}\right) = \infty$.

Get $L: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Claim L is 1-1 onto.

Why: L^{-1} is a LFT. $\quad W = \dfrac{az+b}{cz+d}$

solve for z $\longrightarrow$ $czw + dw = az + b$

$$z = \frac{dz-b}{-cz+a}$$

Just like $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Fact: $L^{-1}(w) = \dfrac{dw-b}{-cw+a}$

Big Fact: An LFT that fixes 3 points must be the identity.

Why: $\dfrac{az_j+b}{cz_j+d} = z_j \quad \leftarrow \quad L(z_j) = z_j \quad j = 1,2,3$

"fixed"

$$\boxed{cz_j^2 + (d-a)z_j - b = 0}$$

Quadratic equation with 3 distinct roots!

Aha! Must be the zero poly. $\quad c = 0$

$d - a = 0 \leftarrow \frac{a}{d} = 1$

$b = 0$

$$L(z) = \frac{az+0}{0z+d} = \frac{a}{d}z$$

$= z \leftarrow$ identity map. ✓

Consequence: An LFT is determined by 3 points.

Why: L_1, L_2 send 3 pts to same 3 points.

$L_1 \circ L_2^{-1}$ fixes 3 pts. So $\equiv$ identity

So $L_1 \equiv L_2$.

Good news: Circles are determined by 3 points.

EX: $\dfrac{z-i}{z+i}$