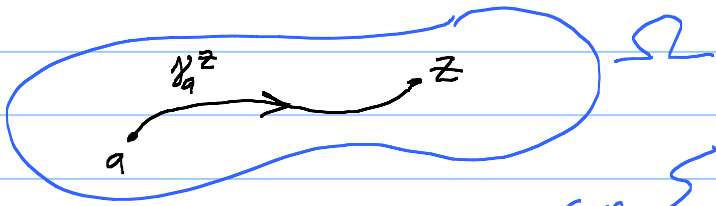
Lesson 39 on 18.1, 18.2 Potential Theory & Conformal Mapping

HWK 12: 36, 37, 38

[WebEx Off. Hr.
Tues. 8-9 pm

Recall: Given a harmonic fcn u on a simply connected domain Ω, there is a harmonic conjugate v on Ω such that $f = u + iv$ is analytic on Ω.



C-R $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$v_x = -u_y$
$v_y = u_x$

Want v with $\nabla v = \underbrace{-u_y \hat{i} + u_x \hat{j}}_{\vec{F}}$

$\text{Curl } \vec{F} = \Delta u \hat{k} \equiv 0.$

So v exists and

$$v = \int_{\gamma_a^z} \vec{F} \cdot d\vec{s}$$

Note: Simply Connected was important for path independence.

EX: Holes cause problems: $u = \text{Ln}|z| = \text{Ln}(x^2+y^2)^{1/2}$

$\Omega = \mathbb{C} - \{0\}$.

If I had a v for u on Ω, then v would work on $\tilde{\Omega}$.

Ouch! $v = \text{Arg} z + C$ on $\tilde{\Omega}$. No way $u + iv$ analytic at jump.

$\tilde{\Omega} = \mathbb{C} - (-\infty, 0]$

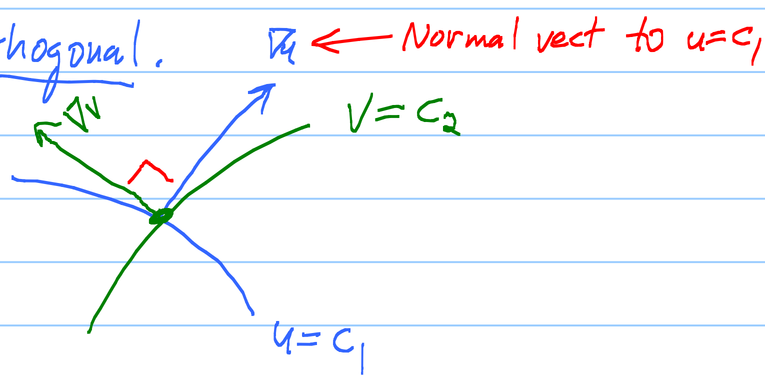
$\text{Ln}|z| + i \text{Arg} z$ analytic on $\tilde{\Omega}$.

Fact: u is harmonic on a domain Ω if and only if $u = \text{Re} f$ where f is analytic on every disc in Ω.

Properties of harmonic fcns.

If $\nabla u = \vec{F}$ and u is harmonic, then u is called a <u>potential fcn</u>. If $\underbrace{u+iv}_{f}$ is analytic, then f is a <u>complex potential</u>.

Big fact: Level sets of u and v in an analytic f are <u>orthogonal</u>.

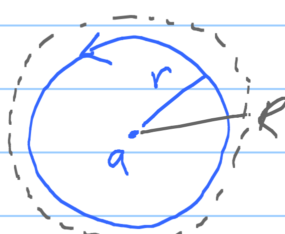


$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \underbrace{\frac{\partial u}{\partial y}}_{= -\frac{\partial v}{\partial x}}\underbrace{\frac{\partial v}{\partial y}}_{= \frac{\partial u}{\partial x}} \equiv 0 \qquad \text{C-R eqns!}$$

u electrostatic potential : $v=c$ are field lines.

u potential for fluid flow: $v=c$ are streamlines

Big Fact 2: Harmonic fcns satisfy the <u>Averaging Property</u>.



u harmonic on $D_R(a)$. Let v be a harm conj for u on $D_R(a)$. $f = u + iv$.

Know $f(a) = \frac{1}{2\pi i}\int_{C_r(a)} \frac{f(z)}{z-a}\,dz$ $\qquad C_r(a): z(t) = a + re^{it},\ 0 \le t \le 2\pi$, $z'(t) = ire^{it}$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{it})}{(\cancel{a}+re^{it})-\cancel{a}}\, ire^{it}\,dt$$

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt$$ ← Averaging Prop for analytic fcns!

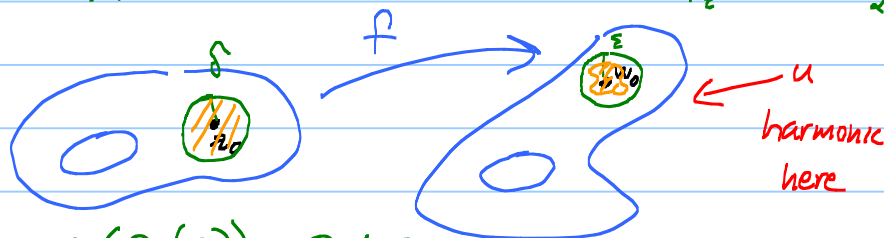
Take the real part.

$$\boxed{u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{it})\,dt}$$ ← Ave Prop for harmonic fcns.

or $$\boxed{u(a) = \frac{1}{2\pi r}\int_0^{2\pi} u(a+re^{it})\,\underbrace{r\,dt}_{ds}}$$

Big Fact 3: Harmonic fcns are preserved under composition with an analytic fcn, i.e., if u is harmonic and f is analytic, then $u \circ f$ is harmonic.

Why:



$f(D_\delta(z_0)) \subset D_\varepsilon(w_0)$ by continuity

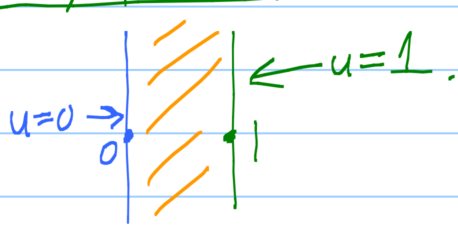
Get harmonic conj for u on $D_\varepsilon(w_0)$. $F = u + iv$

Aha! $F \circ f$ is analytic.

So $u \circ f = \operatorname{Re} F \circ f$ is harmonic.

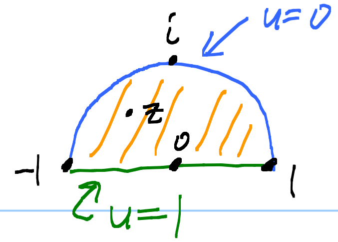
Alternatively: Use chain rule and C-R Eqns.

Easy problem: Find a harmonic fcn that is

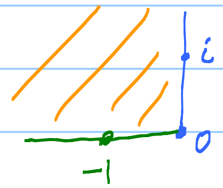


Obvious sol^n: $u(x,y) = x$

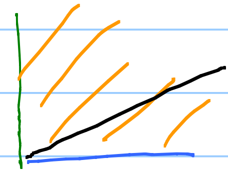
Looks hard:



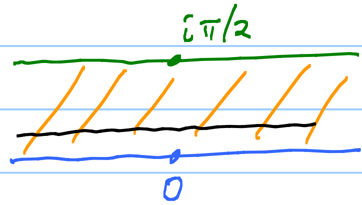
$\xrightarrow{\frac{z-1}{z+1}}$



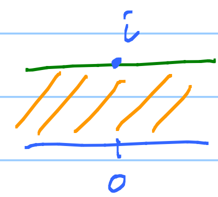
$\xrightarrow{-iz}$ $= e^{-i\pi/2} z$



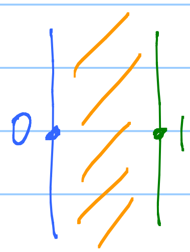
$\xrightarrow{\operatorname{Log} z}$



$\xrightarrow{\frac{2}{\pi} z}$



$\xrightarrow{-iz}$



f = Compositions.

$u = \operatorname{Re} z$

Sol^n to hard problem: $u = \operatorname{Re} f(z)$

$$u = \operatorname{Re}\left[-i\,\frac{2}{\pi}\, \operatorname{Log}\left\{ -i\left(\frac{z-1}{z+1}\right)\right\}\right]$$

$$= \operatorname{Re}\left[-i\,\frac{2}{\pi}\left(\operatorname{Ln}\left|\frac{z-1}{z+1}\right| + i \operatorname{Arg}\left[-i\left(\frac{z-1}{z+1}\right)\right]\right)\right]$$

$$= \frac{2}{\pi} \operatorname{Arg}\left[-i\left(\frac{z-1}{z+1}\right)\right]$$

$z = x + iy$

$\operatorname{Arg} = \operatorname{Tan}^{-1} \frac{v}{u}$

= fcn of x and y.

Prob: