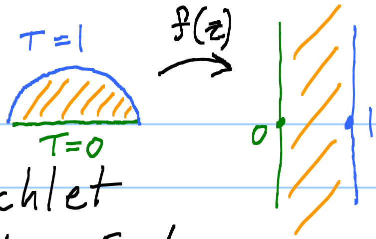
Lesson 40 on 18.3 Heat problems HWK 12: Lessons 36, 37, 38 due tonight

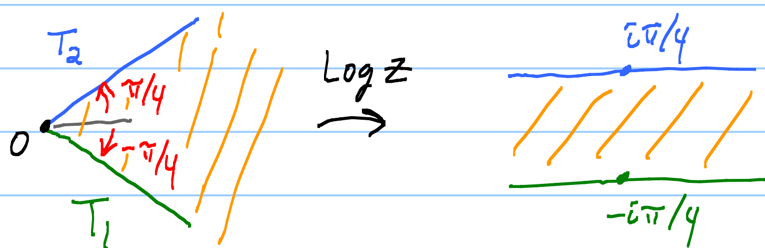


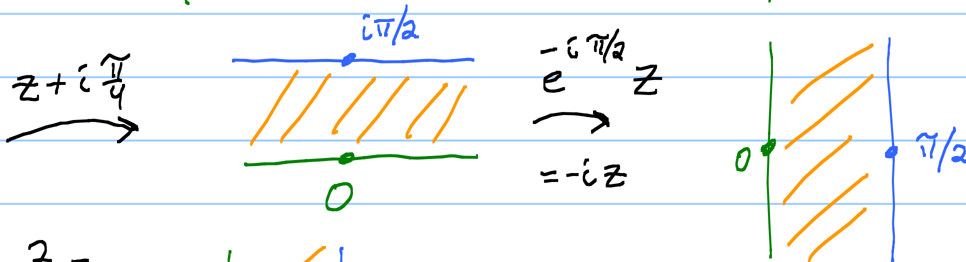
$u = \mathrm{Re}\, z = x$

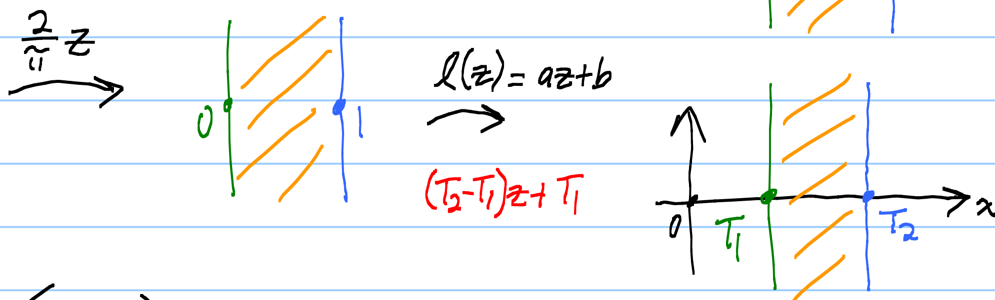
Dirichlet Problem: Find harmonic u in domain with specified boundary values.

$u \circ f$ solves prob on 

$= \mathrm{Re}\,[\underbrace{f(z)}]$ ← complex potential



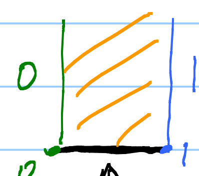




$$\begin{cases} \ell(0) = T_1 \\ \ell(1) = T_2 \end{cases} \qquad \begin{aligned} a\cdot 0 + b &= T_1 \\ a\cdot 1 + b &= T_2 \end{aligned}$$

Re [Composition] solves prob.

EX: Easy problem



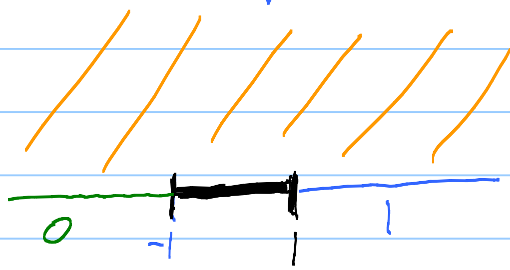
u = temp, $\Delta u \equiv 0$.

Insulated implies $\frac{\partial u}{\partial \hat{n}} = 0$ ($\hat{n}$ = normal dir.)

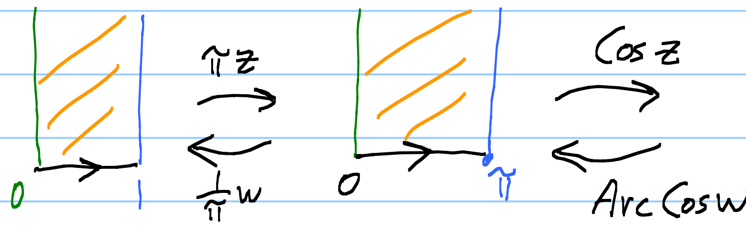
Obvious sol^n: $u = \mathrm{Re}\, z = x$

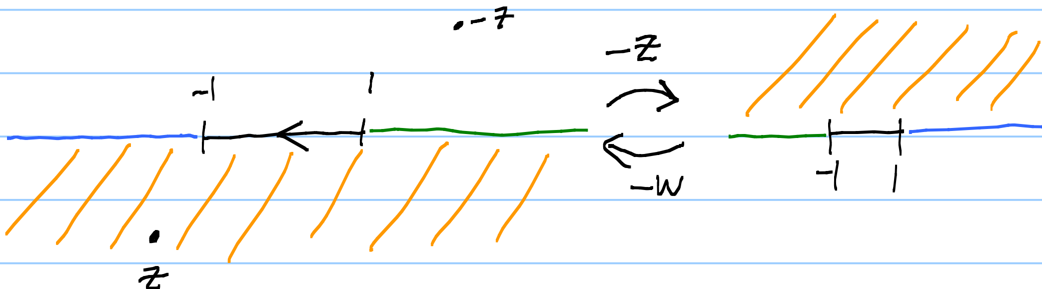
Big observation: Conformal maps (analytic fcns) preserve boundary conditions!

EX: Harder looking problem:



Hmmm.





Answer: $u = \mathrm{Re}\left[\frac{1}{\pi}\,\text{ArcCos}(-w)\right]$

change w to z.

Möbius Transformations

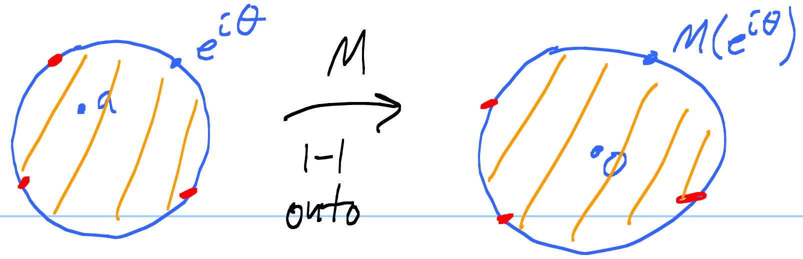
$$M(z) = \frac{z-a}{1-\bar{a}z} \qquad a \in \mathbb{C},\ |a|<1$$

Fact: $M: D_1(0) \to D_1(0)$
1-1, onto, conformal

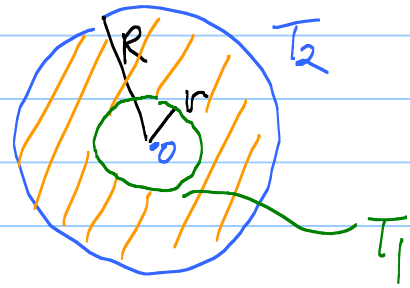
On unit circle: $z = e^{i\theta}$

$$\left|M(e^{i\theta})\right| = \left|\frac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}} \cdot \frac{e^{-i\theta}}{e^{-i\theta}}\right| = \left|\frac{(e^{i\theta}-a)\,e^{-i\theta}}{e^{-i\theta}-\bar{a}}\right| = \frac{|w|}{|\bar{w}|}\,\left|e^{-i\theta}\right| = 1 \cdot 1 = 1$$

Since $e^{-i\theta} = \overline{e^{i\theta}}$ $\qquad \dfrac{e^{i\theta}-a}{e^{-i\theta}-\bar{a}} = \dfrac{w}{\bar{w}}$



Easy problem:



First stab:

$u = \ln|z|$

Harmonic ✓

on $|z| = R$: $u(z) = \ln R$

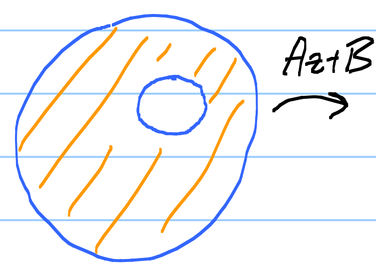
on $|z| = r$: $u(z) = \ln r$

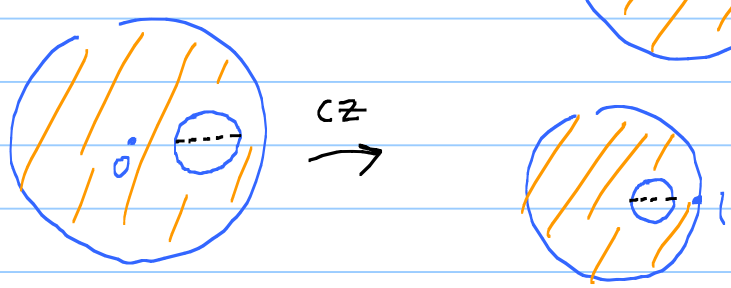
Idea: Compose with $\ell(z) = az + b$.

Want $\begin{cases} \ell(\ln r) = T_1 \\ \ell(\ln R) = T_2 \end{cases}$ $\begin{cases} a\ln r + b = T_1 \\ a\ln R + b = T_2 \end{cases}$

Get a, b. Solution: $\ell(\ln|z|)$.

Harder looking problem:

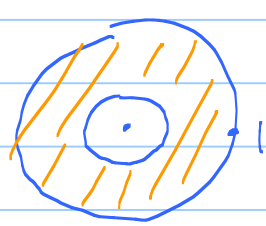




$\xrightarrow{\frac{z-a}{1-\bar{a}z}}$

Take a real.

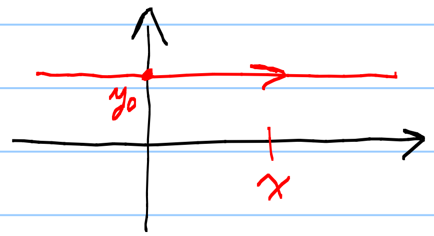
$\bar{a} = a$



Get sol^n by composition from easy case.

Pretty Picture: $w = \sin z$

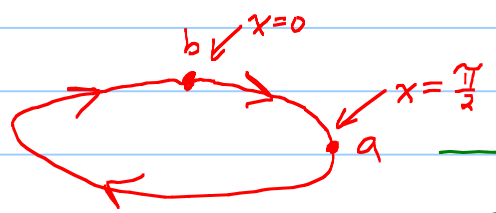
$$u + iv = \sin(x+iy)$$
$$= \sin x \cosh y + i \cos x \sinh y$$

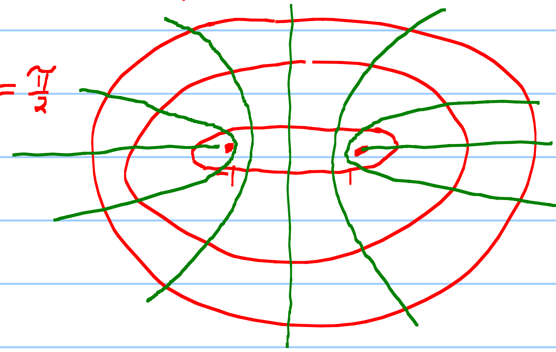


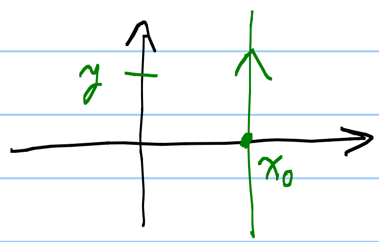
$$\sin x \underbrace{\cosh y_0}_{a} + i \cos x \underbrace{\sinh y_0}_{b}$$

$$u = a \sin x \qquad v = b \cos x$$

Ellipse! $\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 \equiv 1$







$$\underbrace{\sin x_0}_{a} \cosh y + i \underbrace{\cos x_0}_{b} \sinh y$$

$$u = a \cosh y \qquad v = b \sinh y$$

Hyperbola! $\left(\frac{u}{a}\right)^2 - \left(\frac{v}{b}\right)^2 \equiv 1$

$$\left(\cosh^2 y - \sinh^2 y \equiv 1\right)$$

Fact: Harmonic fcns cannot have "hot spots".

Ave Property shows this. Consequence: Maximum Principle. Max occurs on boundary.