Lesson 41 on 18.4 Fluid flow

HWK 13: Lessons 39, 40, 41 due Wed.
WebEx Office Hour Tues 8-9 pm

Lesson 42 on Monday: Read 18.5 on the Poisson formula

<u>Maximum Principle</u>: If u is a non-constant harmonic function on a domain Ω, then u cannot have a local maximum inside Ω. [Local max at z_0 means: There is an $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subset \Omega$ and $u(z_0) \geq u(z)$ for all $z \in D_\varepsilon(z_0)$.]

<u>And</u>, if Ω is bounded and u is continuous up to the boundary, then the max of u occurs on the boundary.

<u>Consequence</u>: If solves the Dirichlet prob:

$$\begin{cases} \Delta u \equiv 0 & \text{in } \Omega \\ u = \varphi & \text{on boundary} \end{cases}$$

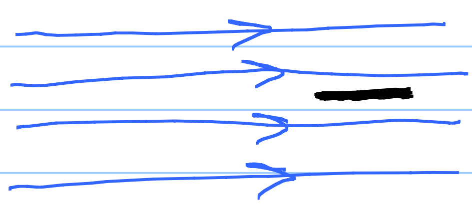
, it is <u>unique</u>.

<u>Why</u>: u_1, u_2 sol'ns, then $u_1 - u_2$ solves $\begin{cases} \Delta u \equiv 0 & \Omega \\ u = 0 & b\Omega \end{cases}$

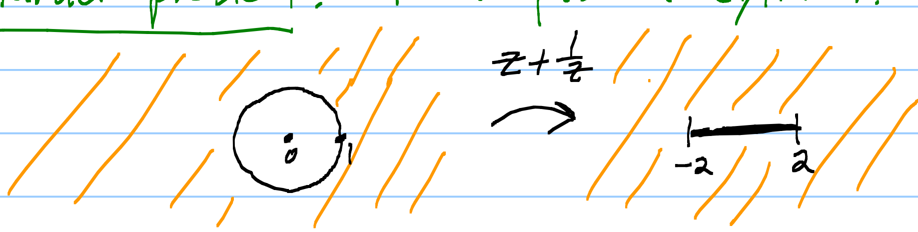
$u_1 - u_2 \leq \text{Max on boundary} = 0$.

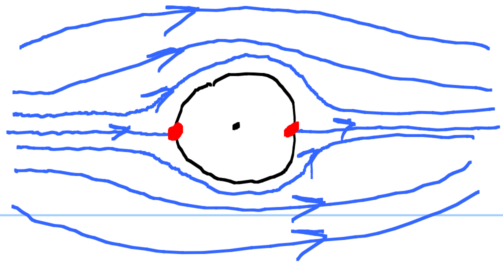
Repeat for $u_2 - u_1$. Get $u_2 - u_1 \leq 0$. So $u_1 \equiv u_2$.

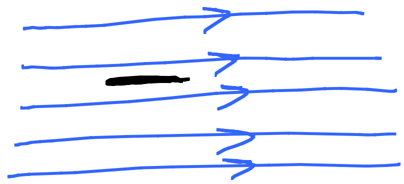
<u>Fluid flow</u>: First: Easy problem: Find flow past a sideways skinny canoe paddle.



<u>Harder problem</u>: Flow past a cylinder.







Incompressible, Irrotational, Inviscid

$\operatorname{div} \vec{V} \equiv 0$, $\operatorname{Curl} \vec{V} \equiv 0$, no friction

Get potential φ for flow. $\vec{V} = \nabla \varphi$

Get a harmonic conjugate for φ. Call it ψ.

$F(z) = \varphi + i\psi$ is analytic. F is a complex potential.

Note: $\varphi = c$ ← equipotential curves

$\psi = c$ ← streamlines!

Important fact: $\vec{V} = V_1 \hat{i} + V_2 \hat{j}$ velocity.

Complex version $V = V_1 + i V_2$

$$\vec{V} = \nabla \varphi, \qquad \text{So} \quad V = \frac{\partial \varphi}{\partial x} + i \underbrace{\frac{\partial \varphi}{\partial y}}_{-\frac{\partial \psi}{\partial x}}$$

C-R Eqn.

$$V = \frac{\partial \varphi}{\partial x} - i \frac{\partial \psi}{\partial x}$$

$$V = \overline{\left(\frac{\partial \varphi}{\partial x} + i \frac{\partial \psi}{\partial x} \right)} = \overline{F'(z)}$$

$$\boxed{V = \overline{F'(z)}}$$

EX: $F(z) = z + \frac{1}{z}$ $\qquad F'(z) = 1 - \frac{1}{z^2}$

$$V = \overline{F'(z)} = 1 - \frac{1}{\bar{z}^2} = 1 - \frac{1}{\bar{z}^2} \cdot \frac{z^2}{z^2}$$

$$= 1 - \frac{z^2}{|z|^4} = 1 - \frac{(x^2-y^2) + 2xyi}{(x^2+y^2)^2}$$

$$= \underbrace{\left[1 - \frac{(x^2-y^2)}{(x^2+y^2)^2}\right]}_{V_1(x,y)} - \underbrace{\left[\frac{2xy}{(x^2+y^2)^2}\right]}_{V_2(x,y)} i$$

Stagnation points: $F'(z) = 1 - \frac{1}{z^2} = 0$

$z^2 = 1$

$z = \pm 1$

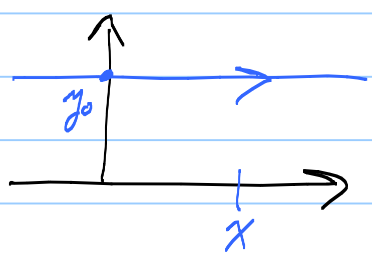
Inverse of Joukovski map: $w = z + \frac{1}{z}$

$zw = z^2 + 1$

$z^2 - zw + 1 = 0$

$$\boxed{z = \frac{w + \sqrt{w^2-4}}{2}}$$

Get streamlines



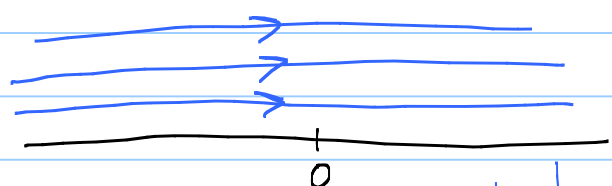
$\psi = c$

$\text{Im}\left(z + \frac{1}{z}\right) = c$ ✓

or $z = \frac{w + \sqrt{w^2-4}}{2}$ with $w = x + iy_0$

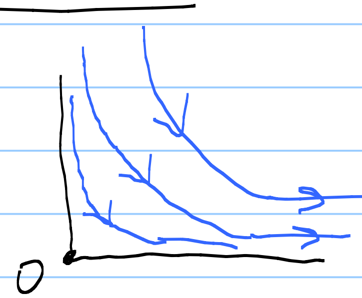
y_0 fixed

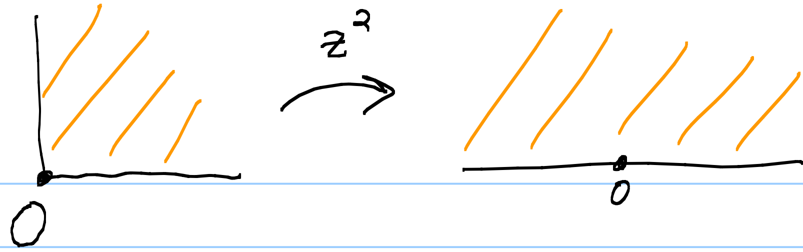
Easy problem: Find flow past a wall.



Harder looking problem:

Flow past a corner.



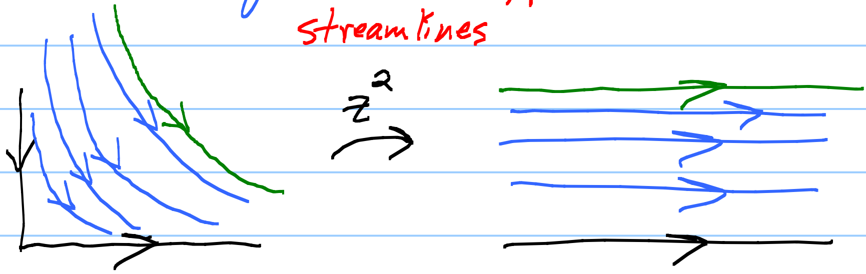


Complex potential: $F(z) = z^2$

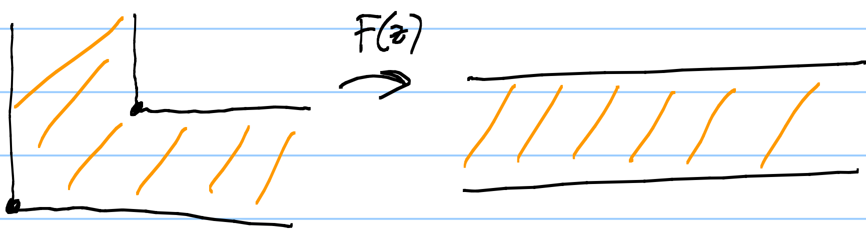
$$= \underbrace{(x^2 - y^2)}_{\varphi} + \underbrace{2xy}_{\psi}\, i$$

$\varphi = \text{Re}\, z^2 = x^2 - y^2 = c$ ← hyperbolas

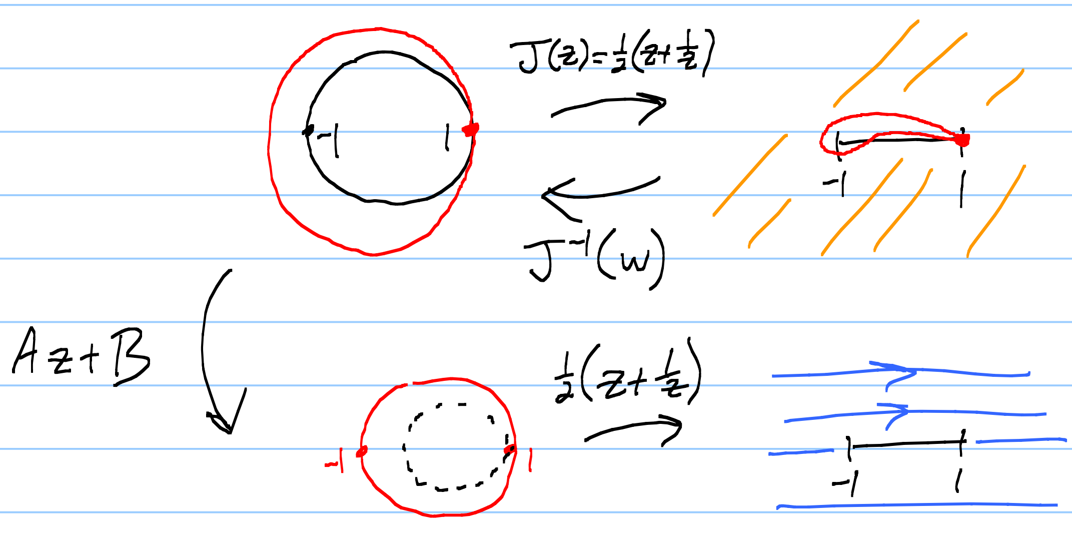
$\psi = \text{Im}\, z^2 = \underbrace{2xy = c}_{\text{streamlines}}$ ← hyperbolas

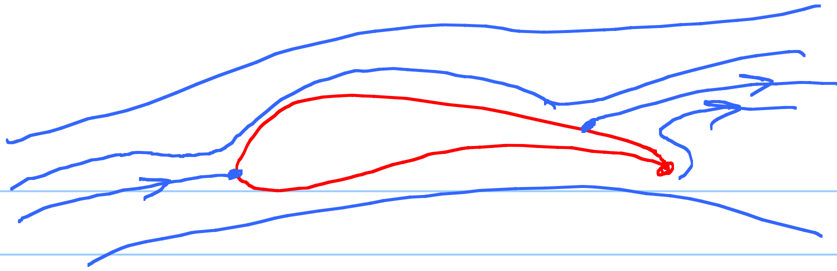


Schwarz-Christofel:

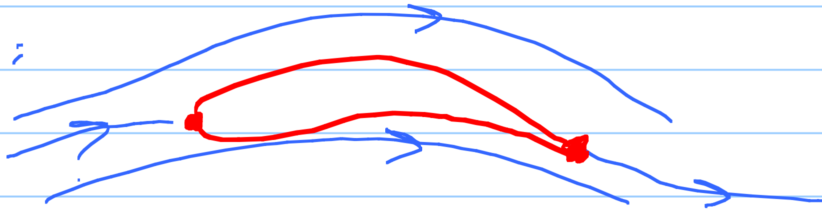


Joukowski Airfoil:





Real wing:



Superimpose spinning cyclinder problem to suck stagnation point back to the tip of the airfoil. Get lift!

(But no drag.)