Lesson 42 on 18.5 Poisson formula

Read 18.5. (No homework from 18.5)

HWK 13: Lessons 39, 40, 41 due Wed.
WebEx Office Hr. Tues 8-9 pm

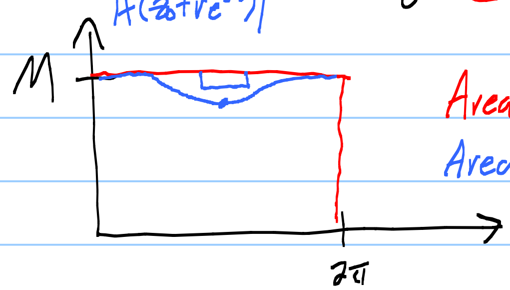
Evaluations please.

Max Principle for analytic fcns: (Maximum Modulus Principle.)

$$\underbrace{|f(z_0)|}_{= M,\ \max} = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0 + re^{it})\,dt\right| \quad \text{Ave Prop}$$

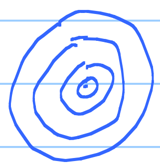
$$\le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0+re^{it})|}_{\le M}\,dt \le M$$

$$< M$$



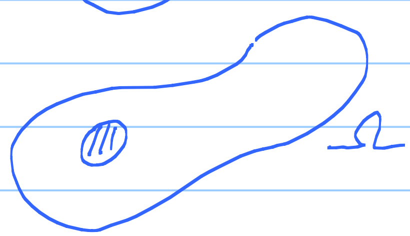
Area $= 2\pi M$

Area $< 2\pi M$



$|f| = M$ on circles. So $|f| = M$ on disc!

$\Longrightarrow f = \text{const.}$ on disc.



$\Longrightarrow f = \text{const}$ on Ω.

Max Mod Thm: If is an analytic non-constant fcn on a domain Ω, then $|f|$ cannot have a local max in Ω.
And, if Ω is bounded and f is continuous up to the boundary, then the max of $|f|$ occurs on the boundary.

Dirichlet Problem on the unit disc:

Formula for sol^n: Want u on $\{z : |z| \le 1\}$

with $\Delta u \equiv 0$ in $D_1(0)$

$u(e^{i\theta}) = \varphi(e^{i\theta}) \leftarrow$ given fcn on boundary.

Say we have sol^{n}.

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta \xrightarrow{r \nearrow 1} \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(e^{i\theta})}_{\varphi(e^{i\theta})}\,d\theta$$

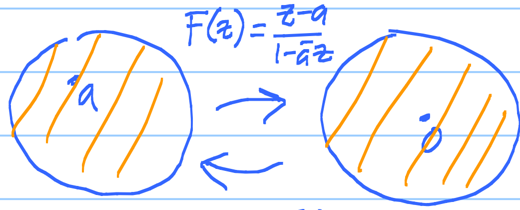
Let $r \nearrow 1$.

$$\boxed{u(0) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta})\,d\theta}$$

Ave of temp on boundary.

Big idea:

$F(z) = \frac{z-a}{1-\bar{a}z}$



$F^{-1} = f(z) = \frac{z+a}{1+\bar{a}z}$

analytic, 1-1, onto (Möbius)

Aha! $u(f(z))$ is harmonic.

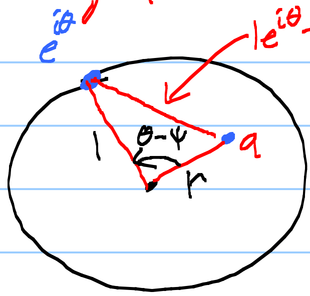
plug in $z=0$

$$u(\underbrace{f(0)}_{a}) = \frac{1}{2\pi}\int_0^{2\pi} u\Big(\underbrace{\frac{e^{it}+a}{1+\bar{a}e^{it}}}_{e^{i\theta}}\Big)\,dt$$

"Just calculus" shows $\quad dt = \dfrac{1-|a|^2}{|e^{i\theta}-a|^2}\,d\theta$

$$\boxed{u(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|a|^2}{|e^{i\theta}-a|^2}\,\varphi(e^{i\theta})\,d\theta}$$

Poisson Integral Formula.



$a = re^{i\psi}$

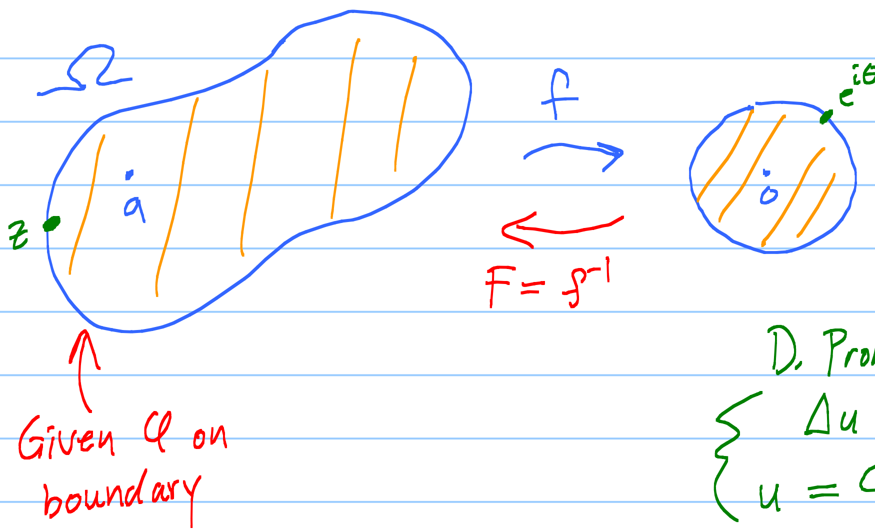
Law of cosines

$$|e^{i\theta}-a|^2 = 1^2 - 2r\cos(\theta-\psi) + r^2$$

$$u(re^{i\psi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}\,\varphi(e^{i\theta})\,d\theta$$

History: Given φ, define u via formula.
Can show u solves D. Prob.

Riemann Mapping Thm: $\Omega \neq \mathbb{C}$, Simply Connected.



f analytic, 1-1, onto

D. Prob on disc:
$$\begin{cases} \Delta u = 0 \\ u = \varphi\circ F \text{ on boundary} \end{cases}$$

Sol^n to D. Prob on Ω: $U = u\circ f$.

$$U(z) = u(f(z)) = u(e^{i\theta})$$
$$= \varphi(F(e^{i\theta}))$$
$$= \varphi(z) \checkmark$$

Another way: $r^n\cos n\theta = \mathrm{Re}(r^n e^{in\theta}) = \mathrm{Re}[z^n]$ ← harmonic
$r^n \sin n\theta = \mathrm{Im}[z^n]$ ← harmonic

Hmmm. Try $u(re^{i\theta}) = \sum_{n=0}^{\infty}(a_n r^n \cos n\theta + b_n r^n \sin n\theta)$ ← harmonic

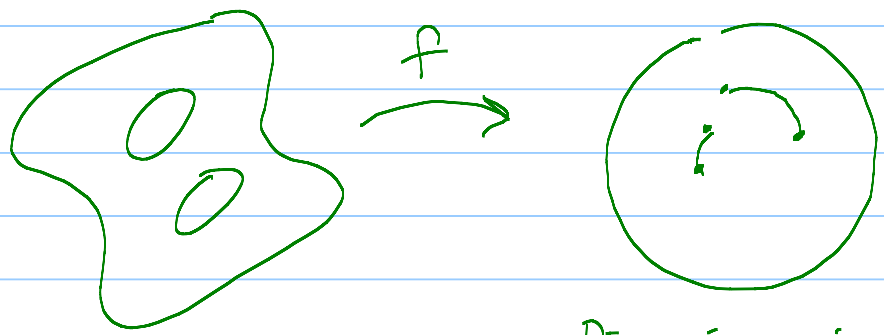
Want $\lim_{r\nearrow 1} u(re^{i\theta}) = \underbrace{\sum_{n=0}^{\infty}(a_n\cos n\theta + b_n \sin n\theta)}_{\text{Fourier Series!}} \overset{\text{want}}{=} \varphi(e^{i\theta})$

$$\left\{ a_0 = \frac{1}{2\pi}\int_0^{2\pi}\varphi(e^{i\theta})\,d\theta \right.$$

$$\begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \varphi(e^{i\theta}) \cos n\theta \, d\theta \\[2ex] b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \varphi(e^{i\theta}) \sin n\theta \, d\theta \end{cases}$$

Write it out, sum geometric series, clean up with trig identities, get Poisson formula.

What if the domain has holes?



Disc minus circular arcs.

Disc is a <u>quadrature domain</u>: f analytic

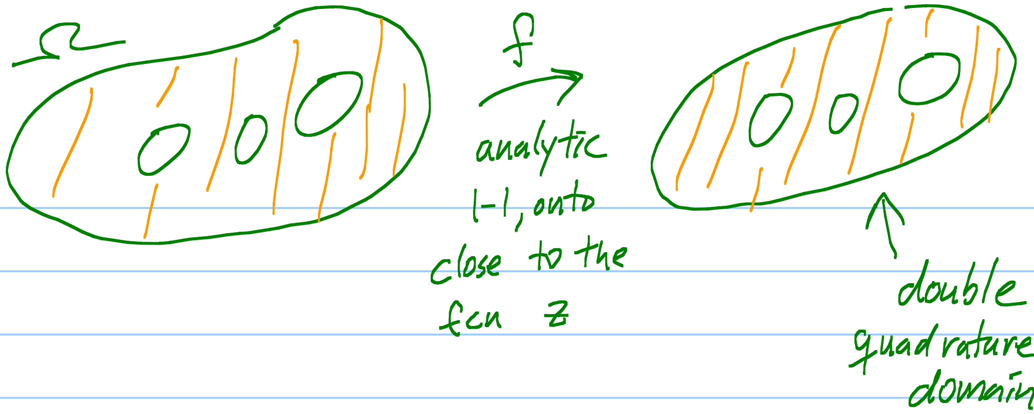
$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, d\theta$$ ← Ave over boundary

$$f(0) = \frac{1}{\pi 1^2}\iint_{D_1(0)} f(z)\, dA$$ ← Ave with respect to area.

Disc a <u>double quadrature</u> domain.

Ω quad domain: $$\iint_\Omega f\, dA = \sum_{j=1}^{N} c_j f(a_j)$$

or $$\int_\gamma f\, ds = \sum_{j=1}^{N} b_j f(z_j)$$



Ω
f
analytic
1-1, onto
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fcn z
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domain