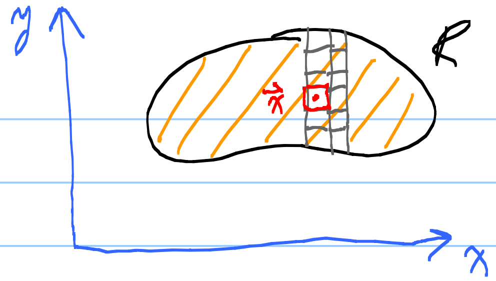
Lesson 5 on 10.3 double integrals   HWK2 Lessons 4,5 due Wed. 1/27



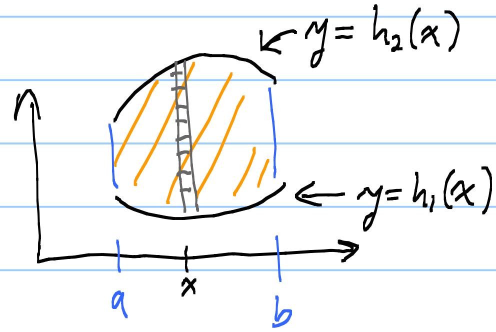
$$\iint_R \rho(x,y)\, dA = \text{Mass of plate}$$

$\uparrow$ density

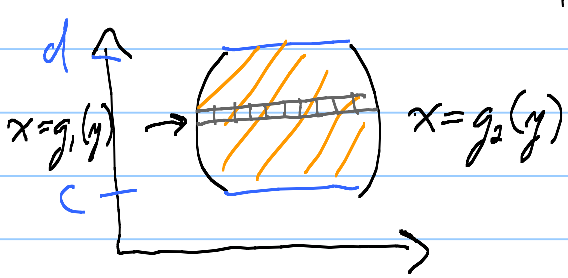
$$\iint_R 1\, dA = \text{Area}(R)$$

$$\iint_R f(x,y)\, dA = \lim_{\Delta x, \Delta y \to 0} \sum f(\vec{x}_{ij})\, \Delta x \Delta y$$

__Iterated integrals:__



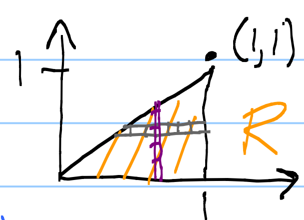
$$\iint_R f\, dA = \int_a^b \left(\int_{h_1(x)}^{h_2(x)} f(x,y)\, dy \right) dx$$



__or__

$$\int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x,y)\, dx \right) dy$$

__EX:__ $\displaystyle\iint_R e^{x^2}\, dA$



$$\int_0^1 \underbrace{\left(\int_y^1 e^{x^2}\, dx \right)}_{\text{Ugh! erf-like. } \int_0^x e^{t^2}\, dt} dy$$

better: $\displaystyle\int_0^1 \Big(\underbrace{\int_0^x e^{x^2}\, dy}_{\left[e^{x^2} y\right]_0^x = xe^{x^2} - e^{x^2}\cdot 0 = xe^{x^2}} \Big)\, dx$

$$= \int_0^1 x e^{x^2}\, dx$$

$u = x^2$

$du = 2x\, dx$

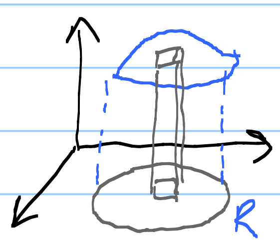
$\boxed{x\,dx = \tfrac{1}{2} du}$

When $x = 0$, $u = 0^2 = 0$,

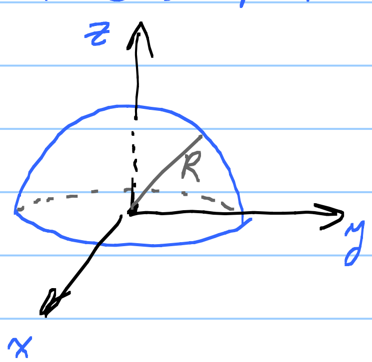
when $x = 1$, $u = 1^2 = 1$

$$= \int_{u=0}^1 e^u \left(\tfrac{1}{2} du\right) = \tfrac{1}{2} e^u \Big]_0^1 = \tfrac{1}{2}(e-1)$$

<u>Volume under $z = f(x,y)$</u>: $\displaystyle V = \iint_R f(x,y)\, dA$

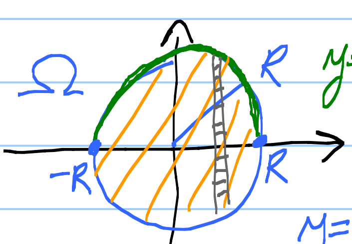


<u>EX</u>: Volume of Northern Hemisphere.



$$x^2 + y^2 + z^2 = R^2$$

$$z = \sqrt{R^2 - x^2 - y^2}$$



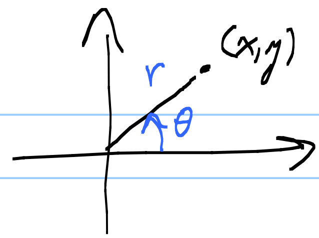
$y = \sqrt{R^2 - x^2}$

$y = -\sqrt{R^2 - x^2}$

$$\iint_\Omega f(x,y)\, dA = \int_{-R}^{R} \left(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2 - x^2 - y^2}\, dy \right) dx$$

Ugh!

Polar coords:

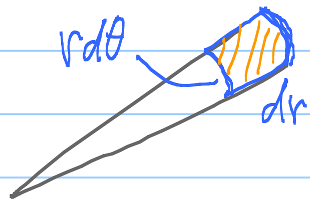


$x = r\cos\theta$

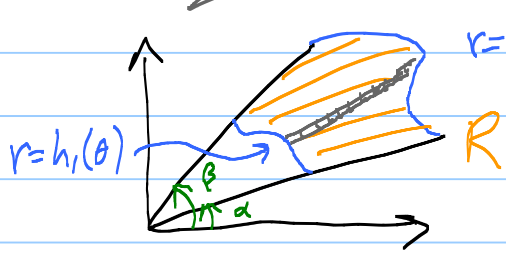
$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\frac{y}{x}$



$dA = r\,dr\,d\theta$



$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \left(\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr \right) d\theta$$

Hemisphere: $$\int_0^{2\pi} \left(\int_0^R \sqrt{\underbrace{R^2 - r^2}_{u}}\; \underline{r\,dr} \right) d\theta$$

When $r=0$, $u=R^2$.

when $r=R$, $u=0$

$du = -2r\,dr$

$\boxed{r\,dr = -\frac{1}{2}du}$

$$\left(\int \quad\right) = \int_{u=R^2}^{0} \sqrt{u}\left(-\tfrac{1}{2}du\right)$$

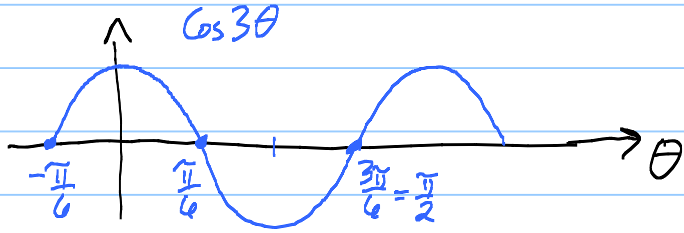
$$= \frac{1}{2}\int_0^{R^2} u^{1/2}\,du = \frac{1}{2}\,\frac{1}{\frac{1}{2}+1}\,u^{\frac{1}{2}+1}\Big|_0^{R^2}$$

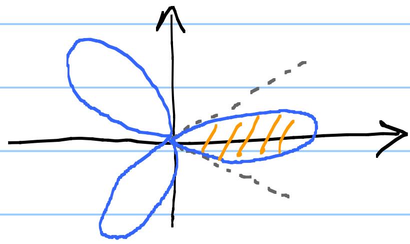
$$= \frac{1}{3}u^{3/2}\Big|_0^{R^2} = \frac{1}{3}R^3$$

$$V = \int_0^{2\pi} \frac{1}{3}R^3\,d\theta = \frac{2}{3}\pi R^3$$

Whole sphere: $\frac{4}{3}\pi R^3$ ✓

Ex: Find the area of one leaf of $r = \cos 3\theta$





$$A = \int_{-\pi/6}^{\pi/6} \underbrace{\left(\int_0^{\cos 3\theta} 1 \cdot r\,dr \right)}_{\frac{1}{2} r^2 \Big|_0^{\cos 3\theta} = \frac{1}{2}\cos^2 3\theta - \frac{1}{2}\cdot 0^2} d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta \, d\theta$$

$\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$

$\sin^2$ $-$

$$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{1}{2}(1 + \cos 6\theta)\, d\theta = \frac{1}{4}\left[\theta + \frac{1}{6}\sin 6\theta\right]_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{4}\left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right) = \frac{1}{4}\cdot\frac{\pi}{3} = \frac{\pi}{12}$$

Centers of mass: $\bar{x} = \dfrac{\sum x_j \Delta m}{\sum \Delta m}$