Lesson 6 on 10.4 Green's Theorem

HWK 2: Lessons 4, 5 due Wed

WebEx Off. Hr. Tues. 8–9 pm

$$\int_\gamma F\,dx + G\,dy = \iint_\Omega \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) \underbrace{dA}_{\substack{dx\,dy \\ dx \wedge dy}}$$

Arrows: Direction bug crawls to make left legs point in to Ω and right legs out. "Positive sense"

How I remember: Stokes' Thm: $\int_\gamma \omega = \iint_\Omega d\omega$

ω differential one-form.

$\omega = F\,dx + G\,dy$

Rules: $dx \wedge dx = 0$

$dy \wedge dy = 0$

$dx \wedge dy = -dy \wedge dx$

$$d(F\,dx + G\,dy) = dF \wedge dx + dG \wedge dy$$

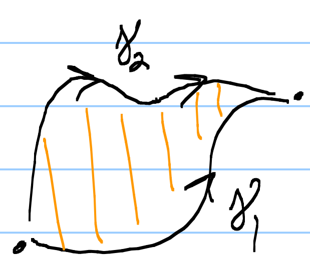
$$= \left(\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy\right) \wedge dx + \left(\frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy\right) \wedge dy$$

$$= 0 + \frac{\partial F}{\partial y} \underbrace{dy \wedge dx}_{= -dx \wedge dy} + \frac{\partial G}{\partial x} dx \wedge dy + 0$$

$$= \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dx \wedge dy \quad \checkmark$$

Notation: $\int_\gamma \vec{F} \cdot d\vec{r} = \iint_\Omega \operatorname{Curl} \vec{F} \cdot \underbrace{d\vec{A}}_{\hat{k}\,dA}$

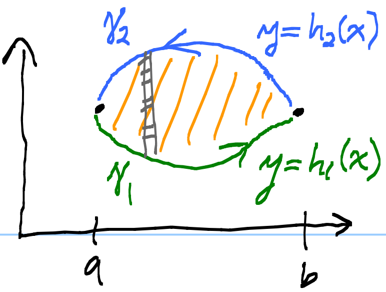
Aha! $\operatorname{Curl} \vec{F} \equiv 0$, then



$\gamma = \gamma_1 \cup (-\gamma_2)$

$$0 = \int_\gamma \vec{F} \cdot d\vec{r} = \int_{\gamma_1} + \int_{-\gamma_2} = \int_{\gamma_1} - \int_{\gamma_2} \longleftarrow \text{IoP}$$

Why Green's works: Convex Ω

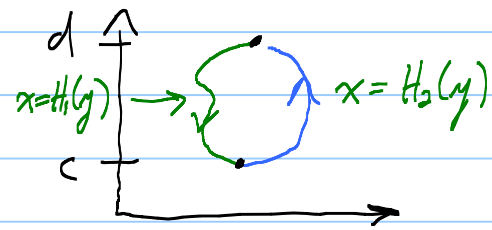


$$\iint_\Omega \frac{\partial F}{\partial y}\,dA = \int_a^b \underbrace{\left(\int_{h_1(x)}^{h_2(x)} \frac{\partial F}{\partial y}\,dy\right)}_{F(x,h_2(x)) - F(x,h_1(x))} dx = -\left(\int_{\gamma_1} + \int_{\gamma_2}\right) F\,dx$$

Fund. Thm. Calc.

$$\gamma_1 : \vec{r}(t) = t\,\hat{\imath} + h_1(t)\,\hat{\jmath}, \qquad a \le t \le b$$

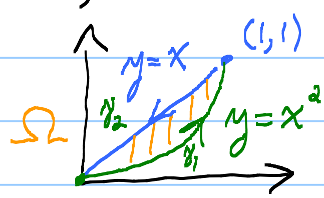
Other half: $\displaystyle\iint_\Omega \frac{\partial G}{\partial x}\,dA =$



Math: Cut up general Ω and add up Green's.

Spivak, Calculus on Manifolds

EX:



$\displaystyle\int_\gamma \vec{F}\cdot d\vec{r}$ where $\vec{F} = e^{x}\hat{\imath} + xy\,\hat{\jmath}$

$\gamma_1 : \vec{r}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath}, \quad 0 \le t \le 1 \qquad \vec{r}\,'(t) = 1\cdot\hat{\imath} + 2t\,\hat{\jmath}$

$-\gamma_2 : \vec{r}(t) = t\,\hat{\imath} + t\,\hat{\jmath}, \quad 0 \le t \le 1 \qquad \vec{r}\,'(t) = \hat{\imath} + \hat{\jmath}$

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} = \int_{\gamma_1} - \int_{-\gamma_2}$$

$$= \int_0^1 \left[e^t\,\hat{\imath} + t\cdot t^2\,\hat{\jmath}\right]\cdot\left[1\cdot\hat{\imath} + 2t\,\hat{\jmath}\right] dt$$

$$- \int_0^1 \left[e^t\,\hat{\imath} + t\cdot t\,\hat{\jmath}\right]\cdot\left[\hat{\imath} + \hat{\jmath}\right] dt$$

$$= \int_0^1 2t^4 - t^2\,dt = \frac{2}{5} - \frac{1}{3} = \frac{1}{15}$$

Verify Green's $\overset{?}{=} \iint_\Omega \frac{\partial}{\partial x}(\underset{G}{xy}) - \frac{\partial}{\partial y}(\underset{F}{e^x})\, dx\,dy$

$$\int_0^1 \left(\int_{x^2}^{x} y \; dy \right) dx$$

$$= \int_0^1 \left[\frac{1}{2} y^2 \right]_{x^2}^{x} dx = \frac{1}{2}\int_0^1 x^2 - x^4 \, dx$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5} \right) = \frac{1}{15} \checkmark$$

Fantastic fact: $\iint_\Omega \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dA = \int_\gamma F dx + G dy$

$G = x \qquad F = -y$

$$2 \, \text{Area}(\Omega) = \int_\gamma -y\,dx + x\,dy$$

$$\boxed{\text{Area}(\Omega) = \frac{1}{2} \int_\gamma -y\,dx + x\,dy}$$

Isoperimetric inequality: $\pi r^2 = \frac{(2\pi r)^2}{4\pi}$

$$\text{Area}(\Omega) \leq \frac{1}{4\pi} \left(\text{Boundary length} \right)^2$$

equality only if circle.

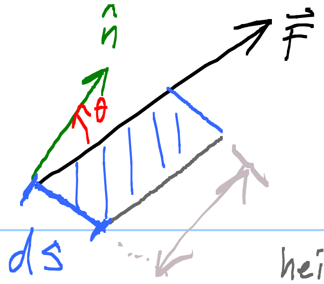
In polar coords: $x = r\cos\theta \qquad dx = \cos\theta\,dr - r\sin\theta\,d\theta$

$y = r\sin\theta \qquad dy = \sin\theta\,dr + r\cos\theta\,d\theta$

$$-y\,dx + x\,dy = r^2\,d\theta$$

$$\boxed{A = \frac{1}{2}\int_\gamma r^2\,d\theta}$$

Flux:

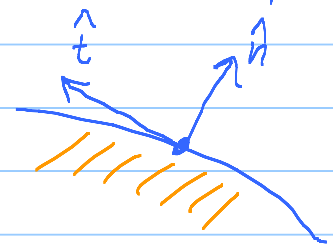


height $= \|\vec{F}\| \cos\theta \, \Delta t$

base $= ds$

$$\text{Flux} = \frac{\text{Outflow}}{\Delta t} = \vec{F} \cdot \hat{n} \, ds$$

Parameterize γ with respect to arc length s.



$$\hat{t} = \frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\jmath}$$

$$\|\hat{t}\| = 1$$

$$\hat{n} = \frac{dy}{ds}\hat{\imath} - \frac{dx}{ds}\hat{\jmath}$$

Note: $\|\hat{n}\| = \|\hat{t}\| = 1$.

$$\text{Flux} = \int_\gamma \vec{F} \cdot \hat{n} \, ds = \int_\gamma (F_1 \hat{\imath} + F_2 \hat{\jmath}) \cdot \left(\frac{dy}{ds}\hat{\imath} - \frac{dx}{ds}\hat{\jmath}\right) ds$$

$$= \int_\gamma F_1 \, dy - F_2 \, dx \underset{\text{Green's}}{=} \iint_\Omega \frac{\partial F_1}{\partial x} - \left(-\frac{\partial F_2}{\partial y}\right) dA$$

$$\text{Total Flux} = \iint_\Omega \operatorname{Div} \vec{F} \, dA \longleftarrow \text{Divergence Thm!}$$

Special case $\vec{F} = \nabla \varphi$.

$$\text{Flux} \quad \int_\gamma \nabla\varphi \cdot \hat{n} \, ds = \iint_{\Omega} \underbrace{\operatorname{Div} \cdot \nabla\varphi}_{\Delta\varphi} \, dA$$

Ω ← Control volume