Lesson 7 on 10.5 Surfaces

HWK 2: Lessons 4,5 due tonight (6,7,8 due next)

\> plot3d({ sqrt(1-y), sqrt(1-z) }, y=0..2, z=0..2);

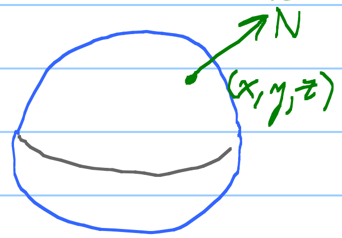
Ways to describe a surface S:

1) Sol^n to an eqn:

$$x^2+y^2+z^2=25$$ ← sphere radius 5

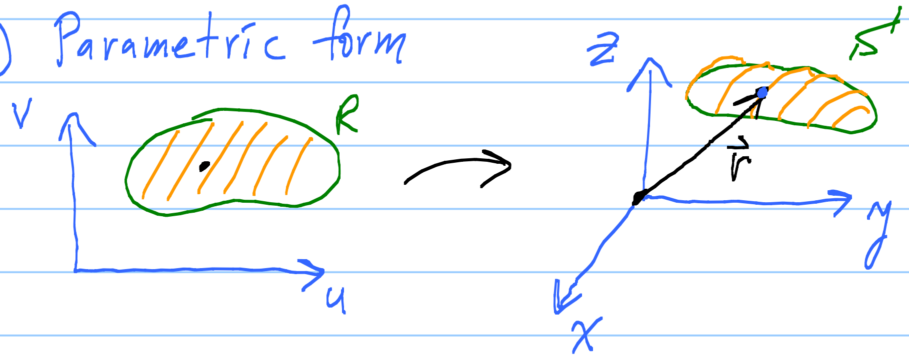
Level set of $x^2+y^2+z^2 = f(x,y,z)$

Normal vector to S: $\nabla f = \underbrace{2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k}}_{\vec{N}}$



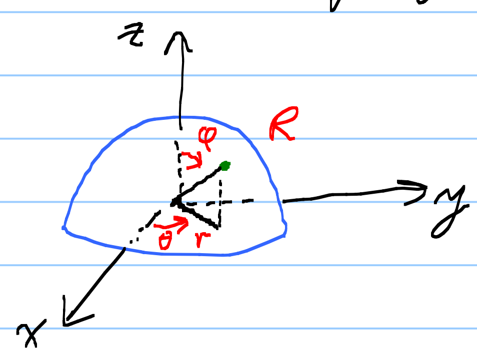
Unit normal vector: $\hat{n} = \dfrac{\vec{N}}{\|\vec{N}\|}$

2) Parametric form



$$\vec{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}$$

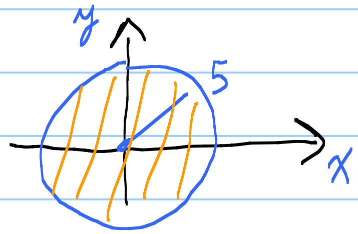
<u>EX</u>: Spherical coords



$$x = (R\sin\varphi)\cos\theta$$
$$y = (R\sin\varphi)\sin\theta$$
$$z = R\cos\varphi$$
$$0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi$$

3) As a graph of a fcn. $z = f(x,y)$

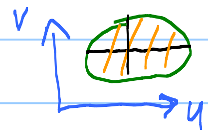
Sphere $z = \underbrace{\sqrt{25 - x^2 - y^2}}_{f(x,y)}$ ← Top half

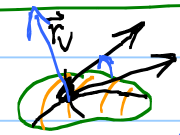


$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$

How to find Normal vector to $\vec{N}$ to S:

Key :





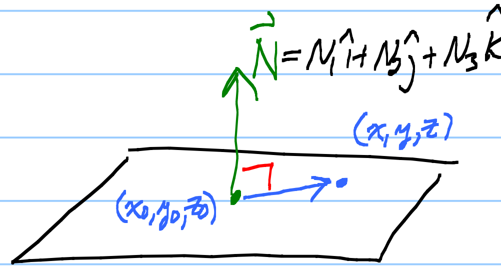
$\vec{r}_u = \frac{\partial}{\partial u}\vec{r}(u,v)$

($\vec{r}_u, \vec{r}_v$ are both tang. vectors)

Aha! $\vec{r}_u \times \vec{r}_v$ is $\perp$ both.

$$\boxed{\vec{N} = \vec{r}_u \times \vec{r}_v}$$

Eqn of a tangent plane:



$$\vec{N} \cdot (\vec{x} - \vec{x}_0) = 0$$

$$\boxed{N_1(x - x_0) + N_2(y - y_0) + N_3(z - z_0) = 0}$$

EX: Cone: 1) $x^2 + y^2 = z^2$

Level set: $\underbrace{x^2 + y^2 - z^2}_{f(x,y,z)} = 0$

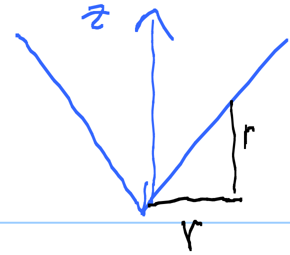
$\vec{N} = \nabla f$

2) $z = \sqrt{x^2 + y^2}$ $\vec{r} = x\hat{i} + y\hat{j} + \sqrt{x^2 + y^2}\,\hat{k}$

+ Top half.

$\vec{N} = \vec{r}_x \times \vec{r}_y$

3) Parametrically



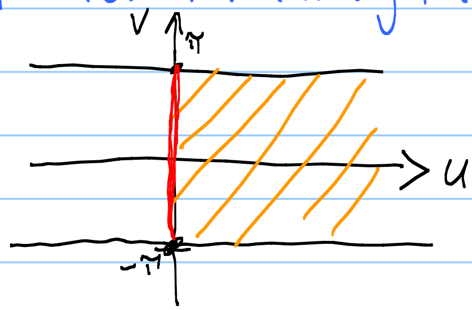
$z = r$

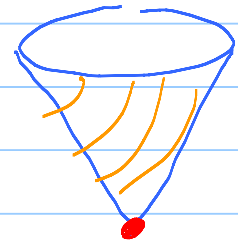
$x = r\cos\theta$

$y = r\sin\theta$

$\vec{r} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r\hat{k}$

$\vec{r} = u\cos v\,\hat{\imath} + u\sin v\,\hat{\jmath} + u\hat{k}$





Prob: Find tangent plane to cone at (3,4,5).

1) $f(x,y,z) = x^2 + y^2 - z^2$

$$\vec{N} = \nabla f\Big|_{(3,4,5)} = 2x\,\hat{\imath} + 2y\,\hat{\jmath} - 2z\,\hat{k}\Big|_{(3,4,5)}$$

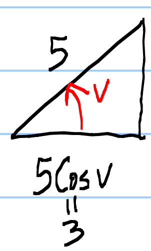
$$= 6\,\hat{\imath} + 8\,\hat{\jmath} - 10\,\hat{k}$$

Tangent plane: $\vec{N}\cdot(\vec{x} - (3,4,5)) = 0$

$$\boxed{6(x-3) + 8(y-4) + (-10)(z-5) = 0}$$

2) Parametrization $\vec{r}(u,v) = \underbrace{u\cos v}_{3}\,\hat{\imath} + \underbrace{u\sin v}_{4}\,\hat{\jmath} + \underset{\substack{\uparrow \\ 5}}{u}\,\hat{k}$

$u = 5$



$v = \tan^{-1}\frac{4}{3}$

$$\vec{r}_u = \frac{\partial}{\partial u}\vec{r} = \cos v\,\hat{\imath} + \sin v\,\hat{\jmath} + 1\cdot\hat{k}$$

$$\vec{r}_v = u(-\sin v)\,\hat{\imath} + u\cos v\,\hat{\jmath} + 0\cdot\hat{k}$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{bmatrix}$$

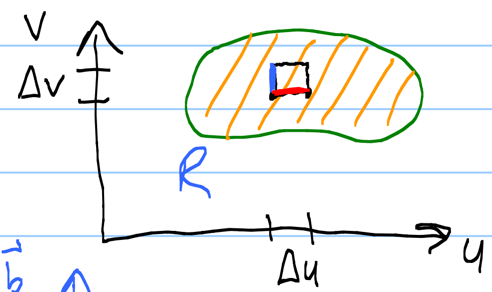
$$= \underbrace{-u\cos v}_{-3}\,\hat{i} - \underbrace{u\sin v}_{-4}\,\hat{j} + \underset{5}{\underset{\uparrow}{u}}\,\hat{k}$$

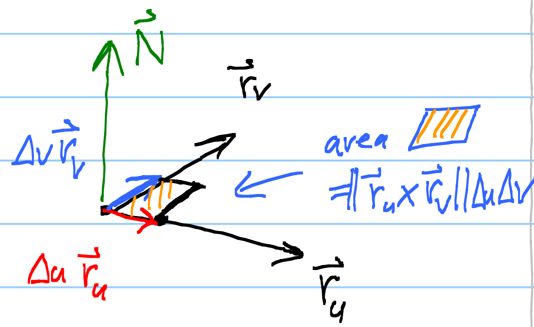
Finally, $u = 5$, $v = \tan^{-1}\frac{4}{3}$

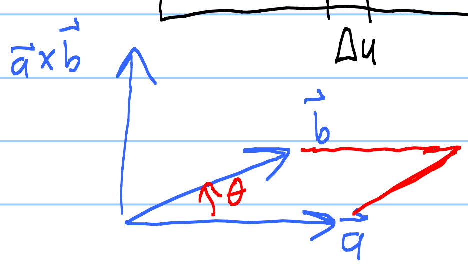
$$\vec{N} = \vec{r}_u \times \vec{r}_v \Big|_{(5, \tan^{-1}\frac{4}{3})} = -3\hat{i} - 4\hat{j} + 5\hat{k}$$

Note: $= -\frac{1}{2}$(previous ans). OK!

Element of Surface area:







Right hand rule

$$\|\vec{a} \times \vec{b}\| = \underbrace{\|\vec{a}\|\|\vec{b}\|\sin\theta}_{\text{area of } \square}$$

$$\text{Surface area} = \iint_R \|\vec{r}_u \times \vec{r}_v\|\, du\, dv$$