Lesson 9 on 10.7 Triple integrals; Divergence Theorem

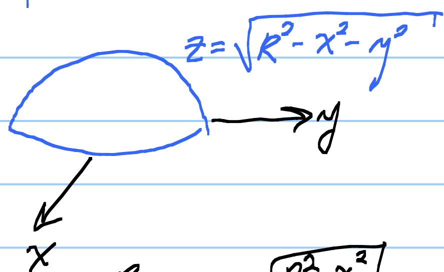
HWK 3: 6, 7, 8 due Wed

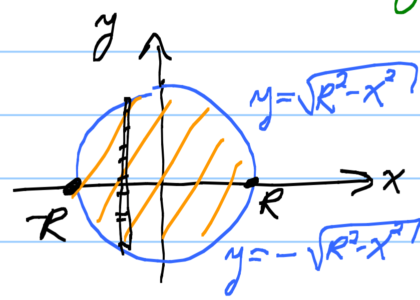
WebEx Off. Hr. Tues 8-9 pm

Old Exam 1 on Home Page

Exam 1, Tues. Feb. 9 on Lessons 1-11.

8-9 pm in EE-129





$$V=\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\sqrt{R^2-x^2-y^2}\,dy\,dx$$

$$=\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{0}^{\sqrt{R^2-x^2-y^2}} 1\,dz\,dy\,dx$$

$-\sqrt{R^2-x^2-y^2}$ $\qquad$ $\rho(x,y,z)$ density fcn.

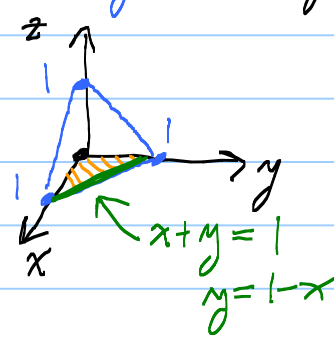
Mass of Earth.

$$\iiint = \lim_{\Delta x,\Delta y,\Delta z\to 0}\sum \rho(x_i,y_j,z_k)\,\Delta x\,\Delta y\,\Delta z$$

EX: Ω region in $\mathbb{R}^3$ in first octant with

$x+y+z\le 1$ and $f(x,y,z)=ye^{x+z}=ye^xe^z$

$$\iiint_{\Omega} f\,dV$$



$$=\int_{x=0}^{1}\int_{y=0}^{1-x}\underbrace{\int_{z=0}^{1-x-y} ye^xe^z\,dz}\,dy\,dx$$

$$ye^{x}\left[e^{1-x-y} - e^{0}\right]$$

$$= \int_0^1 \underbrace{\int_0^{1-x} ye^{1-y} - ye^{x}\, dy}_{\left[-(1-y)e^{1-y} - \frac{y^2}{2}e^{x}\right]_0^{1-x}}\, dx$$

$$= \int_0^1 -(2-x)e^{x} - \frac{(1-x)^2}{2}e^{x} + e\, dx$$

$$= \left[\left(-\frac{x^2}{2} + 3x - \frac{11}{2}\right)e^{x}\right]_0^1 + e$$

$$= \frac{11}{2} - 2e$$

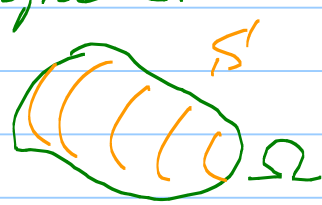
Divergence Theorem: (Gauss' Thm)

$$\iint_S \vec{F}\cdot \underset{\substack{\uparrow \\ \text{outward} \\ \text{pointing}}}{\hat{n}}\, dA = \iiint_\Omega \operatorname{Div}\vec{F}\, \underbrace{dV}_{dx\,dy\,dz}$$

Big application: Heat. Kelvin's Law: Heat flows at a rate and direction proportional to $-\nabla T$.

Calerie = Amount of heat needed to raise one cc of water 1 degree C.

Control Volume:



Heat content: $= c\iiint_\Omega T\, dV$ $\left(\begin{array}{c}\Omega \text{ water} \\ c = 1\end{array}\right)$

Net rate of heat flow out of $S = -k\iint_S \nabla T\cdot\hat{n}\, dA$

Balance rates:

$$-\frac{\partial}{\partial t}\left(c\iiint T\,dV\right) = -k \underbrace{\iint_S \nabla T\cdot\hat{n}\,dA}_{\iiint_\Omega \underbrace{\mathrm{Div}\,\nabla T}_{\Delta T}\,dV}$$

$$\iiint_\Omega c\,\frac{\partial T}{\partial t}\,dV = k\iiint_\Omega \Delta T\,dV$$

$$\iiint_\Omega \left(c\,\frac{\partial T}{dt} - k\,\Delta T\right) dV = 0$$

$$\int_0^{2\pi} \sin x\,dx = 0, \text{ but } \sin x \not\equiv 0.$$

Aha! This holds for any Ω. So

$$\frac{\partial T}{dt} = \frac{k}{c}\,\Delta T \quad \leftarrow \text{Heat Eqn!}$$

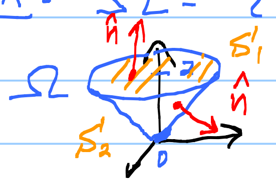
Why: Say $f(x_0,y_0,z_0) = \varepsilon > 0$.

Continuity $\Rightarrow$ $f(x_0,y_0,z_0) > \frac{\varepsilon}{2}$ on some ball B_δ about (x_0,y_0,z_0). Then

$$\iiint_{B_\delta} f\,dV > \frac{\varepsilon}{2}\,\mathrm{Vol}(B_\delta) > 0. \quad ↯$$

Similarly for $-\varepsilon$. If $\iiint_\Omega f\,dV = 0$ for any Ω, f must be zero.

EX: $\Omega = \{(x,y,z) : x^2+y^2 \le z^2,\ 0 \le z \le 2\}$



$$\hat{n} = (\cdots)\,\hat{\imath} + (\cdots)\,\hat{\jmath} + c\,\hat{k}$$

$S' = S_1' \cup S_2'$ — S_1': disc, S_2': cone

$c > 0$ upward
$c < 0$ downward

$\hat{n}_1 = (0,0,1) = \langle 0,0,1\rangle = \hat{k}$

$$S_2' = \{(x,y,z) : z = \underbrace{\sqrt{x^2+y^2}}_{f(x,y)}\}$$

$$\vec{N} = -\frac{\partial f}{\partial x}\hat{\imath} - \frac{\partial f}{\partial y}\hat{\jmath} + 1\cdot\hat{k}$$

$\uparrow$ upward pointing

Want $\vec{N} = f_x\hat{\imath} + f_y\hat{\jmath} - \hat{k}$

$$= \frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\jmath} - \hat{k}$$

Take $\vec{F} = x\hat{\imath} + y\hat{\jmath} + 4z^2\hat{k} = \langle x, y, 4z^2\rangle$

$$\iint_S \vec{F}\cdot\hat{n}\,dA = \iint_R \langle x,y,16\rangle\cdot\langle 0,0,1\rangle\,dA \leftarrow \text{top}$$

$R \leftarrow$ circle of radius 2 in xy-plane

$$+\iint_R \left(x\hat{\imath} + y\hat{\jmath} + 4(x^2+y^2)\hat{k}\right)\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\jmath} - \hat{k}\right)dx\,dy$$

$$\vec{F}\cdot\underbrace{\vec{N}\,dx\,dy}_{\hat{n}\,dA}$$

$$= 16\pi 2^2 + \iint_R \sqrt{x^2+y^2} - 4(x^2+y^2)\,dx\,dy$$

$$= 64\pi + \int_0^{2\pi}\int_0^2 (r - 4r^2)\,r\,dr\,d\theta$$

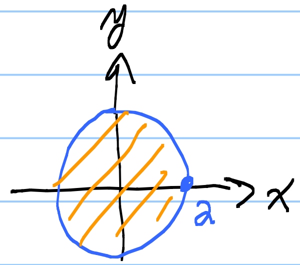
$$= 64\pi + 2\pi\left[\frac{1}{3}r^3 - r^4\right]_0^2$$

$= 64\pi - \frac{80\pi}{3} = \frac{112\pi}{3}$

$\vec{F} = x\hat{i} + y\hat{j} + 4z^2\hat{k}$

Or use Div Thm. $= \iiint_\Omega \text{Div}\,\vec{F}\, dV$

$= \iiint_\Omega 2 + 8z \, dV$



$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=r}^{2} 2+8z \, dz \, r\,dr \, d\theta$

$=$ same thing