MA 528 Exam 1 Spring 2016 Name & Section Key

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1. *(20 pts)* For the surface given by $z = x^3 + y^3$, find a downward pointing normal vector at $(1, 2, 10)$ and an equation for the tangent plane at that point.

$$\varphi(x,y,z) = z - x^3 - y^3 - 1$$

5pts

$$\vec{N} = \nabla\varphi\big|_{(1,2,10)} = -3x^2\hat{i} - 3y^2\hat{j} + \hat{k}\,\Big|_{(1,2,10)} = \underline{-3\hat{i} - 12\hat{j} + \hat{k}}$$

5pts

Downward normal ; $\vec{N} = 3\hat{i} + 12\hat{j} - \hat{k}$

5pts

Equation of tangent plane

$$\underline{3(x-1) + 12(y-2) - (z-10) = 0}$$

5pts

or $\vec{r}(x,y) = x\hat{i} + y\hat{j} + (x^3+y^3)\hat{k}$

$\vec{N} = \vec{r}_x \times \vec{r}_y$ etc.

2. *(20 pts)* Find a potential function for the curl free vector field

$$\vec{F} = \left(\frac{2xy}{1+x^2}\right)\hat{\imath} + \left(2yz + \ln(1+x^2)\right)\hat{\jmath} + (1+y^2)\,\hat{k}.$$

(A): $\frac{\partial \varphi}{\partial x} = \frac{2xy}{(1+x^2)}$ $\quad \varphi = \int \frac{2xy}{\underbrace{(1+x^2)}_{u}}\,dx = \underline{y\,\mathrm{Ln}(1+x^2) + g(y,z)}$ ✓

5 pts

(B): $\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}\left[y\,\mathrm{Ln}(1+x^2) + g(y,z)\right] \overset{\text{want}}{=} 2yz + \mathrm{Ln}(1+x^2)$

$\mathrm{Ln}(1+x^2) + \frac{\partial g}{\partial y} =$

So $\frac{\partial g}{\partial y} = 2yz.$

$g(y,z) = \int 2yz\,dy = \underline{y^2 z + h(z)}$

So $\varphi = y\,\mathrm{Ln}(1+x^2) + y^2 z + h(z)$ 5pts

(C): $\frac{\partial \varphi}{\partial z} = \frac{\partial}{\partial z}\left[y\,\mathrm{Ln}(1+x^2) + y^2 z + h(z)\right]$

5pts

$= 0 + y^2 + h'(z) \overset{\text{want}}{=} 1 + y^2$

$h'(z) = 1$. So $\underline{h(z) = z + C}$

$\varphi = \underline{y\,\mathrm{Ln}(1+x^2) + y^2 z + z + C}$, C arbitrary const.

5 pts. (C=0 ok)

3. *(10 pts)* If γ is the positively oriented boundary curve of an ellipse E of area 7 centered at $(2,3)$, compute

$$\int_\gamma -xy\,dx + x^2\,dy. \underset{\text{Green's}}{=} \iint_E \frac{\partial x^2}{\partial x} - \frac{\partial(-xy)}{\partial y}\,dA$$

5pts

$$= \underbrace{3\iint_E x\,dA}_{2\text{ pts.}} = 3\,\underset{\substack{\text{x-coord of}\\ \underbrace{\text{center of mass}}_{\text{centroid}}}}{\bar{x}} \cdot \text{Area}(E) = 3\cdot 2\cdot 7 = \underline{\underline{42}}$$

3pts.

4. *(10 pts)* Let $\vec{F} = x^2yz\,\hat{\imath} + xy^2z\,\hat{\jmath} + xyz^2\,\hat{k}$. Compute Curl $\vec{F}$ and Div $\vec{F}$.

$$\text{Curl } \vec{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{bmatrix}$$

$$= \hat{\imath}\,(xz^2 - xy^2) - \hat{\jmath}\,(yz^2 - yx^2) + \hat{k}\,(y^2z - x^2z)$$

$$= \underline{x(z^2-y^2)\,\hat{\imath} + y(x^2-z^2)\,\hat{\jmath} + z(y^2-x^2)\,\hat{k}}$$

7pts

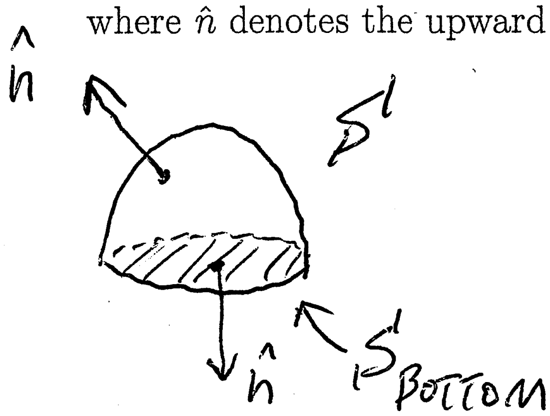
$$\text{Div } \vec{F} = 2xyz + 2xyz + 2xyz = \underline{6xyz}$$

3pts

5. *(20 pts)* Let $\vec{F} = P_1(y)\,\hat{\imath} + P_2(z)\,\hat{\jmath} + z\,\hat{k}$ where $P_1(y)$ is a polynomial in y and $P_2(z)$ is a polynomial in z. Let S denote the surface given by $z = 4 - x^2 - y^2$ above the xy-plane. Compute

$$\int_S \vec{F} \cdot \hat{n}\, dA$$

where $\hat{n}$ denotes the upward pointing unit normal vector to S.



5 pts.

$$\boxed{\text{Div}\,\vec{F} = 0 + 0 + 1 = 1}$$

$$\iint_{S'_{BOTTOM}} \vec{F} \cdot \hat{n}\, dA = \iint \underbrace{[*, *, 0] \cdot (-\hat{k})}_{=0} dA = 0$$

5 pts.

$$\iint_{S'} \vec{F} \cdot \hat{n}\, dA = \left(\iint_{S'} + \iint_{S'_1} \right) \vec{F} \cdot \hat{n}\, dA$$

$$\underset{\substack{\uparrow \\ \text{Div} \\ \text{Thm}}}{=} \iiint_{\Omega} \underbrace{\text{Div}\,\vec{F}}_{1}\, dV = \text{Volume}(\Omega)$$

5 pts

$$= \int_0^{2\pi} \int_0^2 4 - r^2\ r\,dr\,d\theta = 2\pi \int_0^2 4r - r^3\, dr$$

$$= 2\pi \left[2r^2 - \frac{1}{4} r^4 \right]_0^2 = 2\pi(8-4) = \underline{\underline{8\pi}}$$

(See remark at end) 5pts

6. *(20 pts)* Let $\vec{F} = xy\,\hat{\imath} + xz\,\hat{\jmath} + z\,\hat{k}$. Let S denote the surface given by $z = 4 - x^2 - y^2$ above the xy-plane. Write out (but DO NOT COMPUTE) a double integral in the form

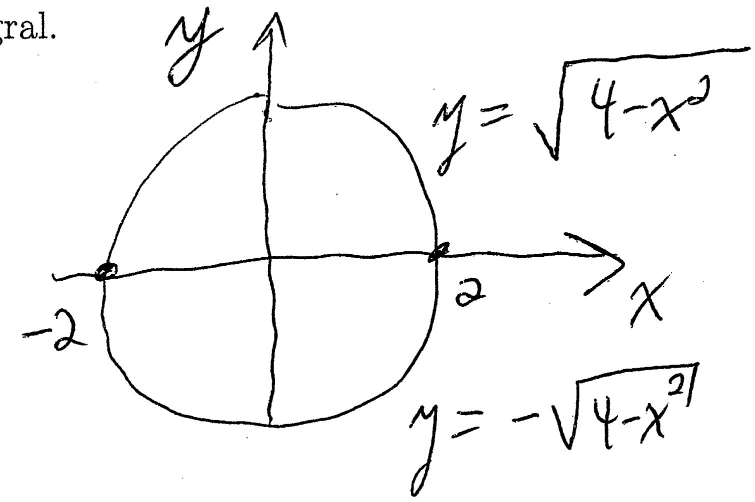
$$\int_{\square}^{\square}\int_{\square}^{\square} \underline{\qquad\qquad 10 \qquad\qquad}\; dy\, dx$$

S S

that yields

$$\int_S \vec{F}\cdot\hat{n}\, dA,$$

where $\hat{n}$ denotes the upward pointing unit normal vector to S. Be sure to put values in all the spots where boxes and blanks appear in the double integral above. Do NOT compute the integral.



$$\vec{r}(x,y) = x\hat{\imath} + y\hat{\jmath} + (4-x^2-y^2)$$

$$\vec{r}_x \times \vec{r}_y = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{bmatrix} = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + \hat{k}$$

upward

$$\iint_S \vec{F}\cdot\hat{n}\, dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[xy\,\hat{\imath} + x(4-x^2-y^2)\hat{\jmath} + (4-x^2-y^2)\hat{k}\right]\cdot [2x, 2y, 1]\; dy\, dx$$

5pts 5pts

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2x^2y + 2xy(4-x^2-y^2) + (4-x^2-y^2)\; dy\, dx$$

10 pts.

5. Alternate correct solution:

$$\iint_{S'} \vec{F} \cdot \hat{n} \, dA =$$

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [P_1(y), P_2(4-x^2-y^2), 4-x^2-y^2] \cdot [2x, 2y, 1] \, dy \, dx$$

10 pts

5pts $\left\{ \begin{array}{l} \iint_R 2x P_1(y) \, dy \, dx = 0 \text{ by symmetry. and odd } x \\ \iint_R 2y P_2(4-x^2-y^2) \, dy \, dx = 0 \text{ by symetry and odd } y \end{array} \right.$

So $\iint_{S'} \vec{F} \cdot \hat{n} \, dA = \iint_R 4-x^2-y^2 \, dy \, dx =$ Volume of gumdrop $= 8\pi$

5 pts.