MA 528 Exam 1 Spring 2016 Name & Section ______________________

Show your work to get credit. Box your answers.

1. *(20 pts)* For the surface given by $z = x^3 + y^3$, find a downward pointing normal vector at $(1, 2, 10)$ and an equation for the tangent plane at that point.

$$\varphi(x,y,z) = z - x^3 - y^3 - 1$$

$$\vec{N} = \nabla\varphi\Big|_{(1,2,10)} = -3x^2\hat{i} - 3y^2\hat{j} + \hat{k}\Big|_{(1,2,10)} = -3\hat{i} - 12\hat{j} + \hat{k}$$

Downward normal ; $\vec{N} = 3\hat{i} + 12\hat{j} - \hat{k}$

Equation of tangent plane

$$3(x-1) + 12(y-2) - (z-10) = 0$$

2. *(20 pts)* Find a potential function for the curl free vector field

$$\vec{F} = \left(\frac{2xy}{1+x^2}\right)\hat{\imath} + \left(2yz + \ln(1+x^2)\right)\hat{\jmath} + (1+y^2)\hat{k}.$$

(A): $\frac{\partial \varphi}{\partial x} = \frac{2xy}{(1+x^2)}$ $\quad \varphi = \int \frac{2xy}{(\underbrace{1+x^2}_{u})}\,dx = y\,\mathrm{Ln}(1+x^2) + \underline{\underline{g(y,z)}}$

(B): $\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}\left[y\,\mathrm{Ln}(1+x^2) + g(y,z)\right] \overset{\text{want}}{=} 2yz + \mathrm{Ln}(1+x^2)$

$\mathrm{Ln}(1+x^2) + \frac{\partial g}{\partial y} \quad =$

So $\frac{\partial g}{\partial y} = 2yz.$

$g(y,z) = \int 2yz\,dy = y^2 z + \underline{\underline{h(z)}}$

So $\varphi = y\,\mathrm{Ln}(1+x^2) + y^2 z + h(z)$

(c): $\frac{\partial \varphi}{\partial z} = \frac{\partial}{\partial z}\left[y\,\mathrm{Ln}(1+x^2) + y^2 z + h(z)\right]$

$= 0 + y^2 + h'(z) \overset{\text{want}}{=} 1 + y^2$

$h'(z) = 1$. So $h(z) = z + C$

$\varphi = y\,\mathrm{Ln}(1+x^2) + y^2 z + z + C$, $\quad C$ arbitrary const.

3. *(10 pts)* If γ is the positively oriented boundary curve of an ellipse E of area 7 centered at $(2,3)$, compute

$$\int_\gamma -xy\,dx + x^2\,dy. \underset{\text{Green's}}{=} \iint_E \frac{\partial x^2}{\partial x} - \frac{\partial(-xy)}{\partial y}\,dA$$

$$= 3\iint_E x\,dA = 3\,\underset{\substack{\text{x-coord of} \\ \underbrace{\text{center of mass}}_{\text{centroid}}}}{\bar{x}} \cdot \text{Area}(E) = 3\cdot 2\cdot 7 = 42$$

4. *(10 pts)* Let $\vec{F} = x^2yz\,\hat{\imath} + xy^2z\,\hat{\jmath} + xyz^2\,\hat{k}$. Compute *Curl* $\vec{F}$ and *Div* $\vec{F}$.

$$\text{Curl } \vec{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{bmatrix}$$

$$= \hat{\imath}\,(xz^2 - xy^2) - \hat{\jmath}\,(yz^2 - yx^2) + \hat{k}\,(y^2z - x^2z)$$

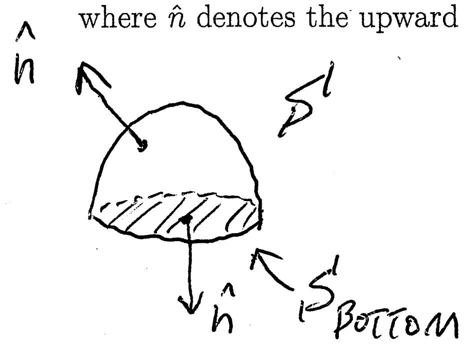
$$= x(z^2 - y^2)\,\hat{\imath} + y(x^2 - z^2)\,\hat{\jmath} + z(y^2 - x^2)\,\hat{k}$$

$$\text{Div } \vec{F} = 2xyz + 2xyz + 2xyz = 6xyz$$

5. *(20 pts)* Let $\vec{F} = P_1(y)\,\hat{\imath} + P_2(z)\,\hat{\jmath} + z\,\hat{k}$ where $P_1(y)$ is a polynomial in y and $P_2(z)$ is a polynomial in z. Let S denote the surface given by $z = 4 - x^2 - y^2$ above the xy-plane. Compute

$$\int_S \vec{F}\cdot\hat{n}\,dA$$

where $\hat{n}$ denotes the upward pointing unit normal vector to S.



$$\text{Div}\,\vec{F} = 0 + 0 + 1 = 1$$

$$\iint_{S'_{BOTTOM}} \vec{F}\cdot\hat{n}\,dA = \iint \underbrace{[*,*,0]\cdot(-\hat{k})}_{=0}\,dA$$

$$= 0$$

$$\iint_{S'} \vec{F}\cdot\hat{n}\,dA = \left(\iint_{S'} + \iint_{S'_1}\right)\vec{F}\cdot\hat{n}\,dA$$

$$\underset{\substack{\text{Div}\\ \text{Thm}}}{=} \iiint_{\Omega} \underbrace{\text{Div}\,\vec{F}}_{1}\,dV = \text{Volume}(\Omega)$$

$$= \int_0^{2\pi}\int_0^2 4 - r^2 \; r\,dr\,d\theta = 2\pi\int_0^2 4r - r^3\,dr$$

$$= 2\pi\left[2r^2 - \frac{1}{4}r^4\right]_0^2 = 2\pi(8-4) = 8\pi$$

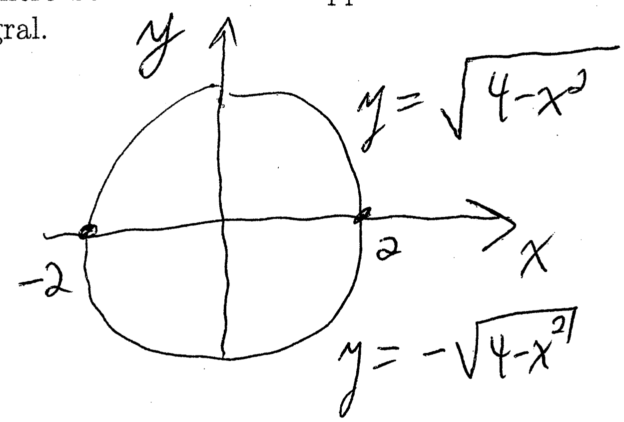
6. *(20 pts)* Let $\vec{F} = xy\,\hat{\imath} + xz\,\hat{\jmath} + z\,\hat{k}$. Let S denote the surface given by $z = 4 - x^2 - y^2$ above the xy-plane. Write out (but DO NOT COMPUTE) a double integral in the form

$$\int_{\square}^{\square}\int_{\square}^{\square} \underline{\hspace{6cm}} \; dy\,dx$$

that yields

$$\int_S \vec{F}\cdot\hat{n}\,dA,$$

where $\hat{n}$ denotes the upward pointing unit normal vector to S. Be sure to put values in all the spots where boxes and blanks appear in the double integral above. Do NOT compute the integral.



$$\vec{r}(x,y) = x\hat{\imath} + y\hat{\jmath} + (4-x^2-y^2)$$

$$\vec{r}_x \times \vec{r}_y = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{bmatrix} = 2x\hat{\imath} + 2y\hat{\jmath} + \hat{k}$$

upward

$$\iint_S \vec{F}\cdot\hat{n}\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[xy\,\hat{\imath} + x(4-x^2-y^2)\hat{\jmath} + (4-x^2-y^2)\hat{k}\right]\cdot [2x, 2y, 1]\;dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2x^2y + 2xy(4-x^2-y^2) + (4-x^2-y^2)\;dy\,dx$$