1. (20) i) Determine all values of $|(i-1)^i|$.

$$(i-1)^i = e^{i \log(-1+i)} =$$

$$= e^{i(\ln\sqrt{2} + i(3\pi/4 + 2n\pi))} = e^{-\left(\frac{3\pi}{4}+2n\pi\right)} e^{i \ln\sqrt{2}}$$

Since $\left|e^{i \ln \sqrt{2}}\right| = 1$, we get

Answer:

$$|(i-1)^i| = e^{-(3\pi/4 + 2n\pi)}, \quad n = 0, \pm 1, \pm 2, \dots$$

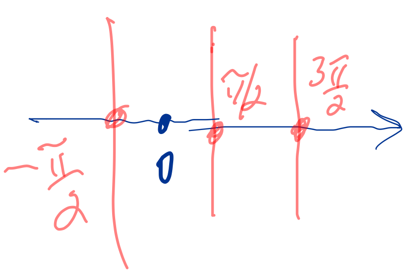
ii) Determine all values z such that the real part of $\cos z$ is 0. (Also draw a graph of the solution set.)

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{ix} e^{-y} + e^{-ix} e^{y}}{2}$$

$$= \frac{1}{2}\left(e^{-y}\cos x + e^{y}\cos(-x)\right) + i(\dots)$$

$$Re = 0 \iff \cos x\,(e^{y} + e^{-y}) = 0$$

$$\iff \cos x = 0$$



Answer:

z : $Re\, z = \frac{\pi}{2} + n\pi$

$n = 0, \pm 1, \pm 2, \dots$

2. (20) Find a harmonic conjugate of $\underbrace{y + e^x \cos y}_{u}$. $\Delta u = 0$ ✓

Want v with $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -1 + e^x \sin y$ (A)

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y$ (B)

(A): $v = \int -1 + e^x \sin y \, dx = -x + e^x \sin y + h(y)$

(B): $\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[-x + e^x \sin y + h(y)\right]$

$= e^x \cos y + h'(y) \overset{\text{want}}{=} e^x \cos y$

So $h'(y) \equiv 0$ and $h(y) = C$, a const.

Answer:

$v = -x + e^x \sin y + C$

3. (20) i) Let Γ be the circle of radius 1 centered at the origin, and traversed once in the counterclockwise direction. Evaluate

$$\int_\Gamma \frac{(e^{z^3} + e^{|z|})}{z} dz. \qquad e^{|z|} = e^1 \text{ on } \Gamma$$

$$= \int_\Gamma \frac{e^{z^3} + e}{z - 0} dz = 2\pi i \left(e^{z^3} + e \right) \Big|_{z=0}$$

$$= 2\pi i \left(e^0 + e \right)$$

Answer:

$2\pi i (1 + e)$

ii) Let L be the line segment from $3 + 3i$ to $1 + i$. Writing your answer in $a + bi$ form, evaluate

$$\int_L \overline{z}\, dz. \qquad -L : z(t) = (1+i) + (2+2i)t \quad 0 \le t \le 1$$

$$\int_{-L} \overline{z}\, dz = \int_0^1 \left[(1-i) + (2-2i)t \right] (2+2i)\, dt$$

$$= \int_0^1 4 + 8t\, dt = 4 + \frac{1}{2} 8 = 8$$

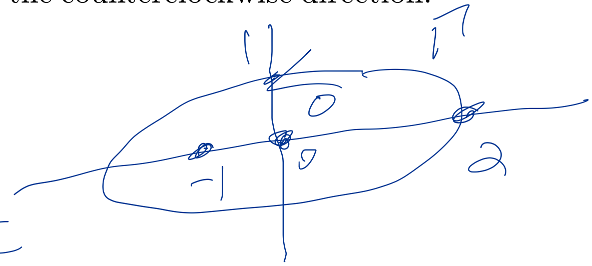
$$\int_L \overline{z}\, dz = -\int_{-L} \overline{z}\, dz =$$

Answer:

-8

4. (20) i) Let Γ be the ellipse $x^2/4+y^2=1$ traversed once in the counterclockwise direction. Evaluate

$$I = \int_\Gamma \frac{\sin(\pi z^2)}{z(z+1)^2}\,dz.$$



$$\frac{1}{z(z+1)^2} = \frac{A}{z} + \frac{B}{(z+1)^2} + \frac{C}{(z+1)}$$

$$1 = A(z+1)^2 + Bz + Cz(z+1)$$

$$1 = (A+C)z^2 + (2A+B+C)z + A$$

$A=1$

$A+C=0, \quad C=-1$

$2A+B+C = 2+B-1 = 0, \quad B=-1$

$$I = \int_\Gamma \sin \pi z^2 \left[\frac{1}{z} - \frac{1}{(z+1)^2} - \frac{1}{(z+1)}\right]$$

$$= 2\pi i\left[\sin \pi 0^2 - 2\pi(-1)\cos \pi(-1)^2 - \sin \pi(-1)^2\right] =$$

Answer:

$-4\pi^2 i$

5. (20) Find the radii of convergence of the following power series.

$$\sum_{n=1}^{\infty} \underbrace{\frac{2^n}{n!}(z-1)^n}_{u_n}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{\frac{2^{n+1}}{(n+1)!}|z-1|^{n+1}}{\frac{2^n}{n!}|z-1|^n} = \frac{2}{n+1}|z-1| \xrightarrow[n\to\infty]{} 0 < 1$$

Converges for all z.

Answer:

$R = \infty$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}(3+4i)^n\, z^{2n}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\frac{[(n+1)!]^2}{[2(n+1)]!}(3+4i)^{n+1} z^{2(n+1)}}{\frac{(n!)^2}{(2n)!}(3+4i)^n z^{2n}}\right| = \frac{(n+1)^2}{(2n+2)(2n+1)}|z|^2$$

$$\to \frac{5}{4}|z|^2 \begin{array}{l} <1 \text{ conv} \\ >1 \text{ div} \end{array} \qquad |z| < \frac{2}{\sqrt{5}} \text{ conv.}$$

Answer:

$R = \frac{2}{\sqrt{5}}$

Problem 1:

a) Find the principal value of $(-2)^{(-i)}$. Express your answer in the form $x + iy$. *[10 points]*

$$= e^{-i\,\mathrm{Log}(-2)} = e^{-i[\mathrm{Ln}\,2 + i\pi]}$$

$$= e^{\pi} e^{-i\,\mathrm{Ln}\,2} = e^{\pi}\cos \mathrm{Ln}\,2 - i e^{\pi} \sin \mathrm{Ln}\,2$$

b) Determine all values of z such that $sin(z) = 3$. *[10 points]*

$$e^{iz} - e^{-iz} = 6i$$

$$(e^{iz})^2 - 6i(e^{iz}) - 1 = 0$$

$$e^{iz} = \frac{6i \pm \sqrt{-36+4}}{2} = 3i \pm i\,2\sqrt{2}$$

$$iz = \log(3 \pm 2\sqrt{2})i = \mathrm{Ln}(3 \pm 2\sqrt{2}) + i\left(\frac{\pi}{2} + 2\pi n\right)$$

$$z = \left(\frac{\pi}{2} + 2n\pi\right) - i\,\mathrm{Ln}(3 \pm 2\sqrt{2})$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Problem 2:

a) For which values of a is the function

$$u(x,y) = e^{\pi x}\cos(ay)$$

harmonic? Here a is supposed to be real **positive** parameter.
[8 points]

$$\Delta u = \pi^2 e^{\pi x} \cos(ay) - a^2 e^{\pi x} \cos(ay)$$

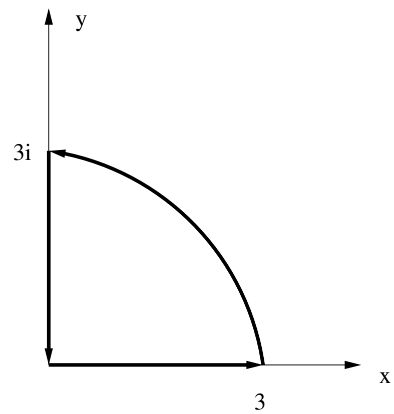
Need $\pi^2 = a^2$. $a = \pm \pi$.

$a > 0 \implies \boxed{a = \pi}$

b) For the choice of the parameter a for which $u(x,y)$ from part a) is harmonic, find a harmonic conjugate $v(x,y)$ of $u(x,y)$, i.e. find a harmonic function $v(x,y)$ such that $f(x,y) = u(x,y) + iv(x,y)$ is analytic.
[12 points]

$$e^{\pi x} \cos \pi y = \operatorname{Re} e^{\pi z}.$$

So $v = e^{\pi x} \sin \pi y$.



Problem 3:

Calculate

$$I = \int_L |z|\, dz$$

on the closed contour L starting at $z = 0$, then going along the real axis to $z = 3$, then following a quarter of a circle with radius 3 to $z = 3i$ and returning to $z = 0$ along the imaginary axis. (see figure above). Is Cauchy's integral theorem valid in this case? Why or why not?

[20 points]

$$I = \int_0^3 t\, dt + \int_0^{\pi/2} |3e^{it}|\, i3e^{it}\, dt$$

$$- \int_0^3 |it|\, i\, dt = \frac{9}{2} + 9\int_0^{\pi/2} i\cos t - \sin t\, dt$$

$$- i\frac{9}{2} = \frac{9}{2} + 9i - 9 - \frac{9}{2}i$$

$$= -\frac{9}{2} + \frac{9}{2}i$$

Problem 4: Let C be the circle with radius 5 centered at $z = i$, oriented counterclockwise. Compute

$$I = \oint_C \frac{z^2}{(z-1)^2(z-i)} dz$$

Please show the details of your work. *[20 points]*

$$\frac{z^2}{(z-1)^2(z-i)} = \frac{A}{(z-1)^2} + \frac{B}{(z-1)} + \frac{C}{z-i}$$

$$z^2 = A(z-i) + B(z-1)(z-i) + C(z-1)^2$$

$$= (B+C)z^2 + (A-(1+i)B-2C)z + (-iA+iB+C)$$

$$\left.\begin{aligned} &B+C=1 \\ &A-(1+i)B-2C=0 \\ &-iA+iB+C=0 \end{aligned}\right\}$$

Ouch! Let's wait and see.

$$I = 2\pi i\,[A\cdot 0 + B\cdot 1 + C\cdot 1]$$

$$= 2\pi i\,(\underbrace{B+C}_{1}) = 2\pi i$$

Problem 5:

a) Find the radius of convergence of the following series. Show the details of your work. *[10 points]*

$$\sum_{n=0}^{\infty} \underbrace{\left(\frac{4-i}{5-2i}\right)^n \frac{n}{2n+1}(z-i)^n}_{u_n} \qquad \left|\frac{u_{n+1}}{u_n}\right| =$$

$$\left|\frac{4-i}{5-2i}\right| \frac{\frac{n+1}{2(n+1)+1}}{\left(\frac{n}{2n+1}\right)} |z-i| \xrightarrow[n\to\infty]{} \frac{\sqrt{17}}{\sqrt{29}} |z-i| \begin{matrix} < 1 \text{ conv} \\ > 1 \text{ div} \end{matrix}$$

$$R = \sqrt{\frac{29}{17}}$$

b) Is the following series convergent or divergent? Give a reason for your answer. *[10 points]*

$$\sum_{n=0}^{\infty} \underbrace{\frac{(3+i)^{(2n+1)}}{(2n)!}}_{u_n} \qquad \left|\frac{u_{n+1}}{u_n}\right| =$$

$$\left| \frac{(3+i)^{[2[n+1]+1]}}{[2(n+1)]!} \Big/ \left[\frac{(3+i)^{2n+1}}{2n!}\right] \right| =$$

$$= |3+i|^3 \frac{1}{(2n+2)(2n+1)} \xrightarrow[n\to\infty]{} 0 < 1$$

Ratio Test $\Longrightarrow$ Converges.