MA 528 Exam 2 Spring 2016 Name & Section Key

Show your work to get credit. Box your answers.

1. *(10 pts)* Find all the complex numbers z such that $e^z = -1 + i$.

$$z = \log(-1+i) = \log\sqrt{2}\, e^{i3\pi/4}$$

$$= \boxed{\ln\sqrt{2} + i\left(\frac{3\pi}{4} + 2n\pi\right)}$$

$$n = 0, \pm 1, \pm 2, \cdots$$

2. *(10 pts)* Show that a real valued analytic function on a domain must be constant.

$$f(x+iy) = u(x,y) + i\underbrace{v(x,y)}_{=0}$$

$$\text{CR-Eqns: } \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \equiv 0 \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \equiv 0 \end{cases} \quad \text{So } \nabla u \equiv 0.$$

$\nabla u \equiv 0$ on a domain $\Rightarrow$ $u = \text{const}$ on domain

3. *(20 pts)* Find a complex conjugate function for the harmonic function

$$u(x, y) = 1 + 4xy + e^{-2y} \sin 2x,$$

i.e., find a function $v(x, y)$ so that $f(x + iy) = u(x, y) + iv(x, y)$ is analytic.

Want v with (A) $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -4x + 2e^{-2y}\sin 2x$

(B) $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4y + 2e^{-2y}\cos 2x$

(A): $v = \int -4x + 2e^{-2y}\sin 2x \, dx$

$= -2x^2 - e^{-2y}\cos 2x + h(y)$

(B): $\frac{\partial v}{\partial y} = 2e^{-2y}\cos 2x + h'(y) \overset{\text{want}}{=} 4y + 2e^{-2y}\cos 2x$

so $h'(y) = 4y$. $h(y) = 2y^2 + C$

so $\boxed{v = -2x^2 + 2y^2 - e^{-2y}\cos 2x + C}$

4. *(10 pts)* Let C be the circle of radius 3 centered at $z = i$ oriented counterclockwise. Compute

$$I = \int_C \frac{z}{(z+5i)(z-i)^2} dz = 2\pi i \, f'(i)$$

where $f(z) = \frac{z}{z+5i}$. $f'(z) = \frac{1(z+5i) - 1 \cdot z}{(z+5i)^2}$

$= \frac{5i}{(z+5i)^2}$. $f'(i) = \frac{5i}{(6i)^2} = \frac{-5}{36} i$.

$I = 2\pi i \left(\frac{-5}{36} i\right) = \boxed{\frac{5\pi}{18}}$

5. *(20 pts)* Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \underbrace{n^3(\pi+i)^n \, (z-2i)^{2n}}_{u_n}$$

$\left|\frac{u_{n+1}}{u_n}\right| = \left| \frac{(n+1)^3 (\pi+i)^{n+1} (z-2i)^{2(n+1)}}{n^3 (\pi+i)^n (z-2i)^{2n}} \right|$

$= \left(\frac{n+1}{n}\right)^3 |\pi+i| \, |z-2i|^2 \to \sqrt{1+\pi^2} \, |z-2i|^2$ <1 conv, >1 div.

$\boxed{R = \frac{1}{(1+\pi^2)^{1/4}}}$

6. *(20 pts)* Suppose γ is a curve that starts at $1-i$ and ends at $1+i$ and stays in the right half plane, i.e., in the set $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Compute

$$\int_\gamma \frac{1}{z}\,dz.$$

$$= \int_\gamma \frac{d}{dz} \log z \; dz = \operatorname{Log}(1+i) - \operatorname{Log}(1-i)$$

$$= \left[\ln\sqrt{2} + i\,\frac{\pi}{4}\right] - \left[\ln\sqrt{2} - i\,\frac{\pi}{4}\right]$$

$$= \boxed{i\,\frac{\pi}{2}}$$

7. *(10 pts)* Is the following series convergent or divergent? Give a reason for your answer.

$$\sum_{n=0}^{\infty} \underbrace{\frac{n}{n+1} e^{in}}_{u_n}$$

$$|u_n| = \frac{n}{n+1}\,|e^{in}| = \frac{n}{n+1} \to 1 \text{ as } n \to \infty.$$

Terms do not $\to 0$ as $n \to \infty$, so series diverges.