NAME SOLUTIONS

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1. Let $\vec{F} = \langle\, 3xy,\ -1,\ 1/(z+1)\,\rangle$ be a force field.

(15) (i) Compute the work done in displacement along the helix given by $\vec{r}(t) = \langle\, \sin t, \cos t, t\,\rangle$ for $0 \le t \le \pi/2$.

$$W = \int_C \vec{F}\cdot d\vec{r}, \quad \vec{r}\,' = \langle \cos t, -\sin t, 1\rangle$$

$$W = \int_0^{\pi/2} \left\langle 3\cos t \sin t, -1, \frac{1}{t+1}\right\rangle \cdot \langle \cos t, -\sin t, 1\rangle\, dt$$

$$= \int_0^{\pi/2} \left(3\cos^2 t \sin t + \sin t + \frac{1}{t+1}\right) dt$$

$$= -\cos^3 t - \cos t + \ln(t+1)\Big|_0^{\pi/2} = 2 + \ln\left(1+\frac{\pi}{2}\right)$$

Answer: $2 + \ln\left(1+\frac{\pi}{2}\right)$

(15) (ii) Compute the work done by the same force $\vec{F}$ in displacement along the line from $\underbrace{(1,2,3)}_{P}$ to $\underbrace{(2,3,3)}_{Q}$.

$$\vec{PQ} = \langle 1, 1, 0\rangle$$

$$C = \{\vec{r} = \langle 1+t, 2+t, 3\rangle, \ 0 \le t \le 1\}$$

$$W = \int_0^1 \left\langle 3(1+t)(2+t), -1, \tfrac{1}{4}\right\rangle \cdot \langle 1, 1, 0\rangle\, dt$$

$$= \int_0^1 \big(3(1+t)(2+t) - 1\big)\, dt = \int_0^1 (3t^2 + 9t + 5)\, dt$$

$$= t^3 + \frac{9}{2}t^2 + 5t\Big|_0^1 = \frac{21}{2}$$

Answer: $21/2$

2. Let $P = (0, 1, 0)$ be a point on a surface S given by $y^4 z = \ln(x + y)$.

(15) (i) Find the upward unit normal $\vec{n}$ to S at P.

$$F(x,y,z) = \ln(x+y) - y^4 z$$

$$F(0,1,0) = 0$$

$$\nabla F = \left\langle \frac{1}{x+y}, \frac{1}{x+y} - 4y^3 z, -y^4 \right\rangle$$

$$\nabla F(0,1,0) = \langle 1, 1, -1 \rangle$$

Upward unit normal $\vec{n} = \frac{1}{\sqrt{3}}\langle -1, -1, 1 \rangle$

Answer: $\frac{1}{\sqrt{3}}\langle -1, -1, 1 \rangle$

(10) (ii) Find the equation of the tangent plane to S at P.

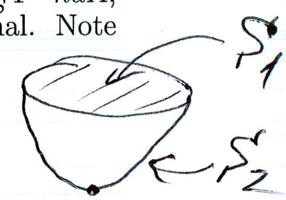
$$\langle 1, 1, -1 \rangle \cdot \langle x, y-1, z \rangle =$$

$$x + (y-1) - z = 0$$

Answer: $x + y - z = 1$

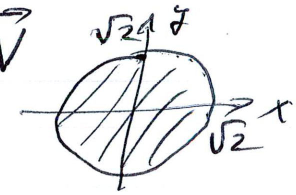
3. Let $\vec{F} = \langle x, y, z \rangle$ and T be the solid region given by $x^2 + y^2 \le z \le 2$.

(15) (i) Without using the divergence theorem, compute the surface integral $\iint_S \vec{F} \cdot \vec{n} dA$, where S is the entire boundary of T, and $\vec{n}$ is the outward unit normal. Note that S is the sum of two smooth surfaces.



$S_1 = \{\vec{r} = \langle u, v, 2 \rangle, u^2 + v^2 \le 2\}$, $\vec{n} = \langle 0, 0, 1 \rangle$

$S_2 = \{\vec{r} = \langle u, v, u^2+v^2 \rangle, u^2+v^2 \le 2\}$



Upward normal $\langle -2u, -2v, 1 \rangle = -\vec{N}$

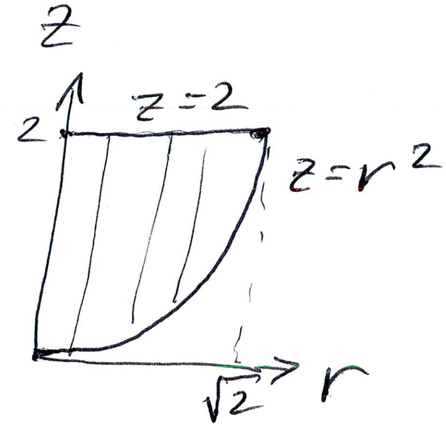
($\vec{N}$ is outward normal)

$\iint_{S_1} \langle u, v, 2 \rangle \cdot \langle 0, 0, 1 \rangle \, du\, dv + \iint_{S_2} \langle u, v, u^2+v^2 \rangle \cdot \langle 2u, 2v, -1 \rangle \, du\, dv$

$= \int_0^{2\pi} \int_0^{\sqrt{2}} (2r + r^3)\, dr\, d\theta = 4\pi + 2\pi$

Answer: 6π

(15) (ii) Using the divergence theorem, set up but do not evaluate, the volume integral corresponding to (i).



$\text{div}\, \vec{F} = 3$

Here $-\vec{N} = \vec{r}_u \times \vec{r}_v$

$= \langle -f_u, -f_v, 1 \rangle$

where $f(u,v) = u^2 + v^2$

$$\int_{0}^{2\pi} \int_{0}^{\sqrt{2}} \int_{r^2}^{2} 3r \; dz\, dr\, d\theta$$

4. (15) Let $\vec{F} = \langle y, 0, 2 \rangle$, and S be the surface given by $z = x + y$ for $0 \le x \le 1$ and $0 \le y \le 1$. Let Cbe the curve bounding S and traversed in the positive direction with respect to the upward unit normal $\vec{n}$ to S. Using Stokes's theorem, compute the line integral $\int_C \vec{F} \cdot d\vec{r}$. (You do not have to compute the integral two ways; only by the area integral in Stokes's theorem).

$$\text{Cur } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & 0 & 2 \end{vmatrix} = \langle 0, 0, -1 \rangle$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \langle -f_u, -f_v, 1 \rangle = \langle -1, -1, 1 \rangle$$

upward normal

Here $\vec{r} = \langle u, v, u+v \rangle$, $f(u,v) = u+v$.

$$\iint_S \text{curl } \vec{F} \cdot \vec{n}\, dA = \int_0^1 \int_0^1 \langle 0, 0, -1 \rangle \cdot \langle -1, -1, 1 \rangle \, du\, dv$$

$$= -1$$

Answer: -1