(12) Review

NAME PRACTICE

1. For the surface given by $z = e^{x\cos y} + 1/(x^2+1)$,

(i) find a normal vector to the surface at (0,0,2),

$$F(x,y,z) = z - e^{x\cos y} - \frac{1}{x^2+1} = 0$$

$$\vec{\nabla} F = \left(-\cos y\, e^{x\cos y} + \frac{2x}{(x^2+1)^2}\right)\vec{i} + x\sin y\, e^{x\cos y}\,\vec{j} + \vec{k}$$

At (0,0,2),

$$\vec{\nabla} F = -\vec{i} + \vec{k}$$

Answer: $-\vec{i} + \vec{k}$

(ii) find the equation of the tangent plane at (0,0,2).

Answer: $-x + z - 2 = 0$

2. For the integral

$$\int_C \left(\frac{2xy}{1+x^2} + z\right) dx + \ln(1+x^2)\, dy + x\, dz,$$

(i) show that the form under the integral sign is exact,

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{2xy}{1+x^2}+z & \ln(1+x^2) & x \end{vmatrix} = (0-0)\vec{i} + (1-1)\vec{j} + \left(\frac{2x}{1+x^2} - \frac{2x}{1+x^2}\right)\vec{k} = \vec{0}$$

and all components have continuous partials.

(ii) evaluate the integral for a curve C which goes from (0,1,2) to (1,0,3).

$$\frac{\partial f}{\partial x} = \frac{2xy}{1+x^2} + z \implies f = y\ln(1+x^2) + zx + g(y,z)$$

Then,

$$\ln(1+x^2) = \frac{\partial f}{\partial y} = \ln(1+x^2) + \frac{\partial g}{\partial y}$$

$\implies g(y,z) = h(z)$. Then,

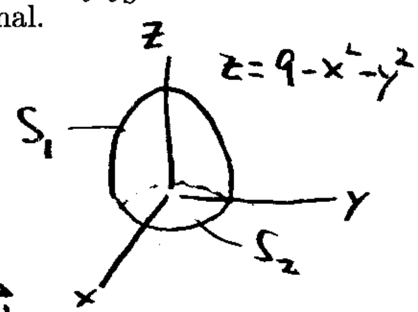
$$x = \partial f/\partial z = x + h'(z) \quad \text{so} \quad h(z) = c$$

$$\int_C = y\ln(1+x^2) + zx \Big|_{(0,1,2)}^{(1,0,3)} = 3 - 0$$

Answer: 3

3. Let $\vec{F} = 3x\vec{i} + 3y\vec{j} + z\vec{k}$ and T be the solid region bounded above by the surface of $z = 9 - x^2 - y^2$, and below by the xy plane.

   (i) Without using the divergence theorem, compute the surface integral $\int\int_S \vec{F}\cdot\vec{n}dA$, where S is the boundary of T, and $\vec{n}$ is the outward unit normal.



$$S_1: \vec{r} = u\vec{i} + v\vec{j} + (9-u^2-v^2)\vec{k} \qquad u^2+v^2 \le 9$$

$$\vec{r}_u = \vec{i} - 2u\vec{k} \qquad \vec{r}_v = \vec{j} - 2v\vec{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = 2u\vec{i} + 2v\vec{j} + \vec{k} = \vec{N}$$

$$S_2: z = 0 \qquad r = u\vec{i} + v\vec{j} \qquad \vec{r}_u \times \vec{r}_v = k = -\vec{N}$$

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_{u^2+v^2\le 9} \left(3u\vec{i} + 3v\vec{j} + (9-u^2-v^2)\vec{k}\right)\cdot\left(2u\vec{i} + 2v\vec{j} + \vec{k}\right) du\,dv$$

$$+ \iint_{u^2+v^2\le 9} \underbrace{\cancel{(3u\vec{i} + 3v\vec{j})\cdot(-\vec{k})}}_{0}\, du\,dv$$

$$= \iint_{u^2+v^2\le 9} 6u^2 + 6v^2 + 9 - u^2 - v^2\, du\,dv = \iint_{u^2+v^2\le 9} (9 + 5u^2 + 5v^2)\, du\,dv$$

$$= \int_0^{2\pi}\int_0^3 (9 + 5r^2)\, r\, dr\, d\theta = \int_0^{2\pi} \frac{9}{2}r^2 + \frac{5}{4}r^4 \Big|_0^3 d\theta = 2\pi\left(\frac{81}{2} + \frac{405}{4}\right)$$

$$= \pi\left(\frac{567}{2}\right)$$

Answer:

$$\frac{567}{2}\pi$$

(ii) Use the divergence theorem to recompute the integral in (i) as a volume integral.

$$\operatorname{div} \vec{F} = 3+3+1 = 7$$

$$\int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} 7 \, r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_6^{3} 7(9-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{63r^2}{2} - \frac{7}{4}r^4 \Big|_0^3 \, d\theta = 2\pi\left(\frac{567}{2} - \frac{567}{4}\right)$$

$$= \frac{567}{2}\pi$$

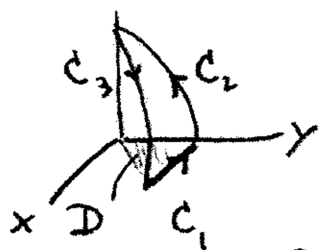
Answer: $\frac{567}{2}\pi$

4. Let S be the surface given by the portion of the graph of $z = 4 - y^2$ cut off by the planes $x = 0$, $z = 0$, and $y = x$. Let $\mathbf{F} = xz\mathbf{j}$.
   (i) Set up, but do not evaluate, the integral $\int\int_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dA$, where $\mathbf{n}$ is the unit upward pointing normal to S.

$$\vec{r} = u\vec{i} + v\vec{j} + (4-v^2)\vec{k} \qquad \vec{r}_u = \vec{i}$$

$$\vec{r}_v = \vec{j} - 2v\vec{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2v \end{vmatrix} = 2v\vec{j} + \vec{k}$$



$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & xz & 0 \end{vmatrix} = -x\vec{i} + z\vec{k}$$

$$\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dA = \iint_D \left(-u\vec{i} + (4-v^2)\vec{k}\right)\cdot\left(2v\vec{j}+\vec{k}\right)\,du\,dv$$

$$\int_{\boxed{0}}^{\boxed{2}} \int_{\boxed{0}}^{\boxed{v}} \boxed{4-v^2}\; du\,dv$$

(ii) Using Stokes's Theorem, evaluate the integral in (i) by means of a line integral.

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} F\cdot d\vec{r}$$

$\vec{F} = xz\,\vec{j}$

$z=0$ on C_1 so $\int_{C_1} \vec{F}\cdot d\vec{r} = 0$

$x=0$ on C_2 so $\int_{C_2} \vec{F}\cdot d\vec{r} = 0$

On C_3, $z = 4-y^2$, $x=y$, $\vec{r}(t) = t\vec{i} + t\vec{j} + (4-t^2)$ $\quad 0 \le t \le 2$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_0^2 t(4-t^2)\,dt = 2t^2 - \frac{t^4}{4}\Big|_0^2$$

$= 8-4$

Answer:

4