Math 528 Exam 2 Name: ____________________

1. (20) i) Determine all values of $|(i-1)^i|$.

$$\left|e^{i\log(i-1)}\right| = \left|e^{i\left(\ln\sqrt{2} + i\left(\frac{3\pi}{4}+2k\pi\right)\right)}\right| \qquad k=0,\pm1,\pm2,\dots$$

$$= e^{-\frac{3\pi}{4}+2k\pi} \qquad k=0,\pm1,\dots$$

Answer:
$$e^{-\frac{3\pi}{4}+2k\pi} \qquad k=0,\pm1,\pm2,\dots$$

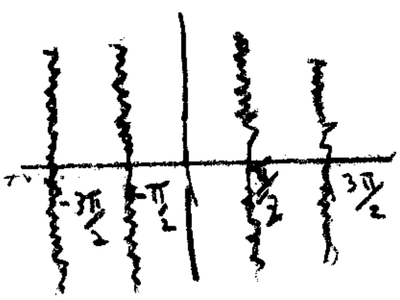
ii) Determine all values z such that the real part of $\cos z$ is 0.

$$\operatorname{Re}\frac{e^{iz}+e^{-iz}}{2}=0 \Rightarrow \operatorname{Re} e^{iz} = -\operatorname{Re} e^{-iz} \Rightarrow$$

$$\operatorname{Re} e^{-y}(\cos x + i\sin x) = -\operatorname{Re} e^{y}(\cos x - i\sin x)$$

$$\Rightarrow \quad e^{-y}\cos x = -e^{y}\cos x$$

$$(e^{y}+e^{-y})\cos x = 0 \qquad e^{y}+e^{-y}\neq 0$$



Answer: $\operatorname{Re} z = \pi/2 + k\pi \quad k=0,\pm1,\dots$

2. (20) Find a harmonic conjugate of $y + e^x \cos y$.

$$u = y + e^x \cos y$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y$$

$$v = e^x \sin y + h(x)$$

$$-1 + \cancel{e^x \sin y} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \cancel{e^x \sin y} + h'(x)$$

$$h(x) = -x + C$$

$$v = e^x \sin y - x$$

$$f(z) = y + e^x \cos y + i(e^x \sin y - x)$$

$$= -iz + e^z$$

Answer:

$$e^x \sin y - x$$

3. (20) i) Let Γ be the circle of radius 1 centered at the origin, and traversed once in the counterclockwise direction. Evaluate

$$\int_\Gamma \frac{(e^{z^3} + e^{|z|})}{z} dz.$$

$$\int_\Gamma \frac{e^{z^3}}{z} dz = 2\pi i e^{z^3} \Big|_{z=0} = 2\pi i$$

$$\int_\Gamma \frac{e^{|z|}}{z} dz = \int_0^{2\pi} \frac{e}{e^{it}} i e^{it} dt = 2\pi e i$$

$z = e^{it}$

$0 \le t \le 2\pi$

Answer:

$2\pi i(1+e)$

ii) Let L be the line segment from $3+3i$ to $1+i$. Writing your answer in $a+bi$ form, evaluate

$$\int_L \bar{z}\, dz.$$

$z = 3+3i+t(-2-2i)$

$0 \le t \le 1$

$\frac{dz}{dt} = -2-2i$

$$\int_0^1 (3-3i + t(-2+2i))(-2-2i) dt$$

$$= -(2+2i)\left((3-3i)t + (-1+i)t^2\right)\Big|_0^1$$

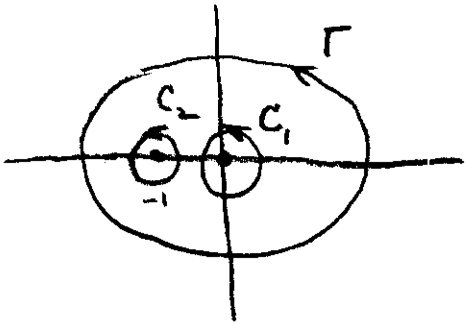
$$= -(2+2i)(3-3i-1+i) = -(2+2i)(2-2i) = -(4+4)$$

Answer:

-8

4. (20) i) Let Γ be the ellipse $x^2/4+y^2=1$ traversed once in the counterclockwise direction. Evaluate

$$\int_\Gamma \frac{\sin \pi z^2}{z(z+1)^2}dz. = \int_{C_1} \frac{\frac{\sin \pi z^2}{(z+1)^2}}{z} dz + \int_{C_2} \frac{\frac{\sin \pi z^2}{z}}{(z+1)^2} dz$$



$$= 2\pi i \left. \frac{\sin \pi z^2}{(z+1)^2} \right|_{z=0} + 2\pi i \left. \left(\frac{\sin \pi z^2}{z} \right)' \right|_{z=-1}$$

$$= 0 + 2\pi i \left. \left(\frac{2\pi \cancel{z} \cos \pi z^2}{\cancel{z}} - \frac{\sin \pi z^2}{z^2} \right) \right|_{z=-1}$$

$$= 2\pi i \left(2\pi(-1) - 0 \right) = -4\pi^2 i$$

Answer:

$-4\pi^2 i$

5. (20) Find the radii of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}(z-1)^n$$

$$\left| \frac{\frac{2^{n+1}(z-1)^{n+1}}{(n+1)!}}{\frac{2^n (z-1)^n}{n!}} \right| = \frac{2}{n+1}|z-1| \to 0 < 1 \text{ for all } z.$$

Answer:

$R = \infty$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}(3+4i)^n z^{2n}$$

$$\left| \frac{\frac{((n+1)!)^2}{(2(n+1))!}(3+4i)^{n+1} z^{2(n+1)}}{\frac{(n!)^2}{(2n)!}(3+4i)^n z^{2n}} \right| = \frac{(n+1)^2}{(2n+2)(2n+1)} |3+4i| |z|^2$$

$$\to \frac{1}{4} \cdot 5 \cdot |z|^2 < 1$$

$$\Rightarrow |z|^2 < 4/5$$

Answer:

$R = 2/\sqrt{5}$