Lecture 4 Power series

Def^{n}s of sequential convergence and series convergence in $\mathbb{C}$ same as in $\mathbb{R}$ with $|\cdot| =$ modulus in place of absolute value.

HWK: $\mathbb{C}$ is complete: Cauchy sequences converge.

Lemma $\sum_{n=1}^{\infty} a_n$ converges, say to $L \in \mathbb{C}$, then $a_n \to 0$ as $n \to \infty$.

Pf $S_N = \sum_1^N a_n$. $a_N = S_N - S_{N-1} \to L - L$ as $N \to \infty$. ✓

Def^n $\sum_1^{\infty} a_n$ is absolutely convergent if $\sum_1^{\infty} |a_n| < \infty$.

Lemma Abs conv series $\Rightarrow$ convergent.

Pf Abs conv $\Rightarrow$ S_N is a Cauchy seq $\Rightarrow$ S_N conv.

Big fact Power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ have a radius of convergence (RoC) $R \geq 0$, meaning

1) $\sum_0^{\infty} a_n (z-z_0)$ converges absolutely in $D_R(z_0)$

2) diverges when $|z-z_0| > R$

3) converges uniformly on $D_r(z_0)$, $0 < r < R$.

Note: $R = 0$ case. Series only converges at $z = z_0$.

$R = \infty$ case. Converges for all z.

Hadamard's formula $\quad \dfrac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

Def^n $\quad \limsup_{n\to\infty} \overset{\in\mathbb{R}}{r_n} = \lim_{N\to\infty} \underbrace{\sup\{r_n : n \geq N\}}_{\text{non-increasing in } N}$

So limit exists (might be $-\infty$).

Note: When $r_n = \sqrt[n]{|a_n|} \geq 0$, $\limsup \geq 0$.

Geometric series $\quad \sum_{n=0}^{\infty} z^n$

$$S_N = 1 + z + \cdots + z^N$$
$$z S_N = \quad z + z^2 + \cdots + z^N + z^{N+1}$$
$$(1-z) S_N = 1 - z^{N+1}$$

$$S_N = \frac{1}{1-z} + \underbrace{\frac{-z^{N+1}}{1-z}}_{E_N(z)}$$

Δ ineq : $\quad |z+w| \leq |z| + |w| \leftarrow$ Numerator estimate

$|z-w| \geq \big||z| - |w|\big| \leftarrow$ Denominator estimate

$|z+w| \geq \big||z| - |w|\big| \swarrow$ (replace w by $-w$ above)

If $|z| > 1$, $\sum z^n$ diverges because $z^n \not\to 0$ as $n \to \infty$.

If $\underline{|z| < r} < 1$, then

$$S_N(z) = \frac{1}{1-z} + E_N(z)$$

Denom est:
$$|1-z| \geq \underbrace{\big|1-|z|\big|}_{=1-|z|}$$

$$|E_N(z)| = \frac{|z^{N+1}|}{|1-z|} \leq \frac{|z^{N+1}|}{1-|z|} = \frac{|z|^{N+1}}{1-|z|} < \frac{r^{N+1}}{1-|z|}$$

$$< \frac{r^{N+1}}{1-r} \quad \text{because } 1-|z| > 1-r.$$

Consequently $S_N(z) \to \frac{1}{1-z}$ uniformly on $D_r(0)$, $0<r<1$.

So $RoC = 1$.

Note: $E_N(x) = \frac{-x^{N+1}}{1-x} \longrightarrow -\infty$ as $x \nearrow 1$.

S_N does not conv uniformly on $D_1(0)$.

Pf of Hadamard's   Case $0 < R < \infty$. Assume $z_0 = 0$.

Pick ρ, r with $|z| < r < \rho < R =$ Hadamard's R

$$\frac{1}{R} < \frac{1}{\rho}$$

$$\sup_{n \geq N} \sqrt[n]{|a_n|}$$

"sliding down to $\frac{1}{R}$"

So $\exists\, N$ such that $\sup_{n \geq N} \sqrt[n]{|a_n|} < \frac{1}{\rho}$

$$\sqrt[n]{|a_n|} < \frac{1}{\rho} \quad \text{when } n \ge N$$

mult by $|z|^n$ $\longrightarrow$ $|a_n| < \frac{1}{\rho^n} \qquad n \ge N$

Hmmm. $|a_n z^n|$: $\qquad |a_n z^n| < \left(\frac{|z|}{\rho}\right)^n$

Aha! Compare tail end of $\sum a_n z^n$ to convergent geom series $\sum \left(\frac{|z|}{\rho}\right)^n$.

Since r, ρ can arbitrarily close to R, see series converges absolutely inside $D_R(0)$.

<u>Get uniform conv in $D_r(0)$</u> $\qquad f(z) = \sum_0^\infty a_n z^n$

$$\left| f(z) - \sum_{n=0}^{M} a_n z^n \right| = \left| \sum_{n=M+1}^{\infty} a_n z^n \right|$$

Take $M \ge N$

$$\le \sum_{n=M+1}^{\infty} |a_n z^n|$$

$$\le \sum_{n=M+1}^{\infty} \left(\frac{|z|}{\rho}\right)^n$$

$$= \frac{\left(|z|/\rho\right)^{M+1}}{1 - \left(\frac{|z|}{\rho}\right)}$$

$$< \frac{(r/\rho)^{M+1}}{1 - \frac{r}{\rho}} \longrightarrow 0 \text{ as } M \to \infty.$$

Unif conv on $D_r(0)$. ✓

<u>Last step</u>: Div if $|z| > R$.

$$\frac{1}{|z|} < \frac{1}{R} \leq \sup_{n \geq N} \sqrt[n]{|a_n|}$$

for all N

So, for each N, there is an $n \geq N$ such that

$$\frac{1}{|z|} < \sqrt[n]{|a_n|}$$

$$\frac{1}{|z|^n} < |a_n|$$

$$1 < |a_n z^n|$$

$a_n z^n$ <u>do not</u> $\to 0$ as $n \to \infty$. Series diverges.

$R=0$, $R=\infty$ is easier than above. Exercise

HWK 1: Prob 7. Easy way to see RoC <u>exists</u>

<u>EX</u>: $\sum_0^\infty n!\, z^n$ $R=0$

<u>EX</u>: $\sum_1^\infty \frac{z^n}{n^2}$ $R=1$. Converges unif on $\overline{D_1(0)}$.

<u>EX</u>: $\sum_0^\infty \frac{z^n}{n!} = e^z$ $R=\infty$.

Stirling's formula

$$\underbrace{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{4n}\right)}$$

$$\frac{1}{\sqrt[n]{n!}} < \frac{1}{(2\pi n)^{1/2n}\left(\frac{n}{e}\right)}$$

$$\longrightarrow 0 \text{ as } n\to\infty.$$

So $\frac{1}{R}$ "=" 0 and $R=\infty$.

Much easier to use Ratio test from freshman calc to see $\sum \frac{z^n}{n!}$ converges for all z.