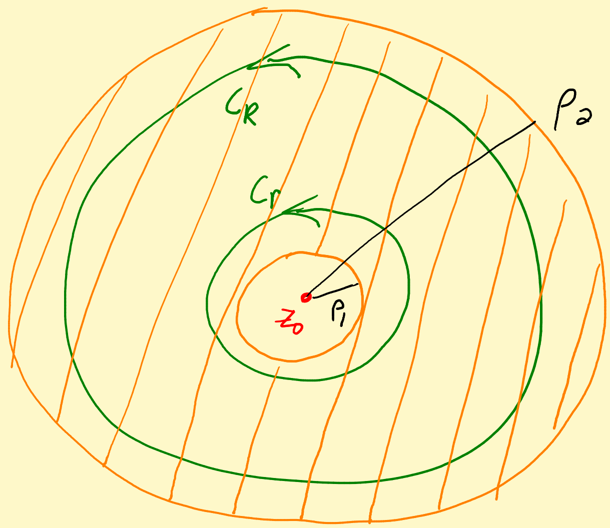
Lecture 27 Laurent series



Annulus

$\rho_1 < r < R < \rho_2$

f analytic on

$A(\rho_1, \rho_2, z_0) = \{z : \rho_1 < |z - z_0| < \rho_2\}$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{} + \underbrace{\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}}_{}$$

converges absolutely in $D_{\rho_2}(z_0)$ and uniformly in $D_R(z_0)$

converges absolutely in $\{z : |z-z_0| > \rho_1\}$ and uniformly in $\{z : |z-z_0| > r\}$

$$a_n = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad , \quad n \in \mathbb{Z} \quad , \quad \text{indep of } \rho : \rho_1 < \rho < \rho_2 .$$

Def^n $a_{-1} = \frac{1}{2\pi i} \int_{C_\rho} f(z)\, dz$ is the residue of f at z_0

$= \operatorname{Res}_{z_0} f$ when z_0 is an isolated sing of f

Theorem 1 Suppose f has an isolated sing at z_0.

1) $a_n = 0$ for $n = -1, -2, -3, \dots \iff z_0$ is removable

2) Only finitely many $a_{-n} \neq 0$ with $n \in \mathbb{N}$ $\iff$ z_0 is a pole

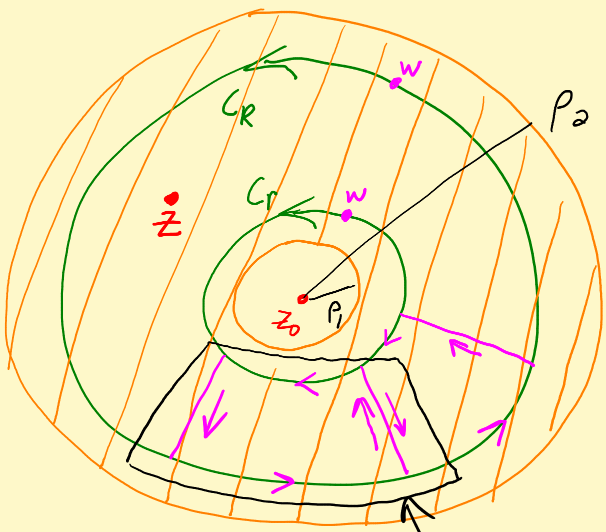
3) Infinitely many $a_{-n} \neq 0$ with $n \in \mathbb{N}$ $\iff$ z_0 is essential

Thm 2 : f has an analytic antiderivative on $D_{\rho_2}(z_0) - \{z_0\}$

$$\iff \operatorname{Res}_{z_0} f = 0 .$$

Pf uses

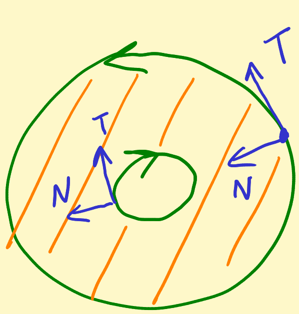
Cauchy theorem for an annulus and the Cauchy integral for an annulus.



f analytic on ▨

$$\left(\int_{C_R} - \int_{C_r} \right) f \, dz = 0$$

Remark



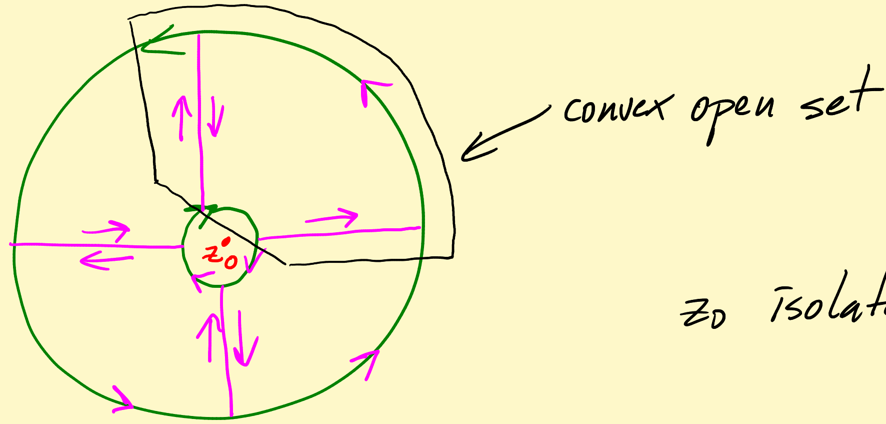
Inward pointing normal N is to left of T

$$\left(\int_{C_R} + \int_{-C_r} \right) f \, dz = 0$$

"Standard orientation" of bndry of annulus.

Pf of Cauchy thm: add up $\int_{\text{pie pieces}}$. Cuts cancel.
Use Cauchy on convex.

EX



z_0 isolated sing

Cauchy integral formula: Apply Cauchy thm to DQ =

$$\frac{f(w)-f(z)}{w-z} \longrightarrow f'(z) \quad \text{as} \quad w \to z.$$

It has a removable sing at z =

$$F(w) = \begin{cases} f'(z) & w = z \\ \dfrac{f(w)-f(z)}{w-z} & w \neq z \end{cases} \quad \text{analytic}$$

So $\left(\int_{C_R} - \int_{C_r}\right) F\, dw = 0$

$$\left(\int_{C_R} - \int_{C_r}\right) \frac{f(w)}{w-z}\, dw - \underbrace{\left(\int_{C_R} - \int_{C_r}\right) \frac{f(z)}{w-z}\, dw}_{f(z)\left[\underbrace{\int_{C_R} \frac{1}{w-z}\, dw}_{2\pi i} - \underbrace{\int_{C_r} \frac{1}{w-z}\, dw}_{=0 \text{ by Cauchy's}}\right]}$$

Cauchy integral formula: $\displaystyle f(z) = \frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right) \frac{f(w)}{w-z}\, dw$

Pf of Laurent. Can assume $z_0 = 0$.

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \underbrace{\frac{f(w)}{w-z}}_{\frac{f(w)}{w}\,\underbrace{\frac{1}{1-\frac{z}{w}}}_{1+\left(\frac{z}{w}\right)+\cdots+\left(\frac{z}{w}\right)^N + E_N\left(\frac{z}{w}\right)}} dw \;-\; \frac{1}{2\pi i}\int_{C_r} \underbrace{\frac{f(w)}{w-z}}_{-\frac{f(w)}{z}\,\underbrace{\frac{1}{1-\frac{w}{z}}}_{1+\left(\frac{w}{z}\right)+\cdots+\left(\frac{w}{z}\right)^N + E_N\left(\frac{w}{z}\right)}} dw$$

$$= \sum_{n=0}^{N} \left(\frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w^{n+1}}\, dw \right) z^n + \mathcal{E}_N(z)$$

$$E_N\left(\frac{w}{z}\right) = \frac{\left(\frac{w}{z}\right)^{N+1}}{1-\frac{w}{z}}$$

$$\sum_{n=1}^{N+1} \left(\frac{1}{2\pi i}\int_{C_r} f(w)\, w^{n-1}\, dw \right) \frac{1}{z^n} + \mathcal{E}_N(z)$$

Where

$$|\mathcal{E}_N(z)| \le \frac{1}{2\pi}\,\frac{1}{|z|}\left(\max_{C_r} |f|\right) \frac{\left(\frac{r}{|z|}\right)^{N+1}}{1-\frac{r}{|z|}}$$

$\to 0$ as $N \to \infty$ uniformly when $|z| > \tilde{r} > r$.

Remark

$$\int_{C_r} f\, dw = \int_{C_\rho} f\, dw = \int_{C_R} f\, dw$$

$r < \rho < R$ via Cauchy thm on an annulus.

Pf of Thm 1) ($\Rightarrow$) Suppose $a_{-n} = 0$, $n \in \mathbb{N}$.

Then f = convergent power series on $D_{\rho_2}(z_0) - \{z_0\}$.

So z_0 removable.

($\Leftarrow$) Assume z_0 removable. Then

$$a_{-n} = \underbrace{\frac{1}{2\pi i}\int_{C_\rho} \frac{f(w)}{(w-z_0)^{(-n+1)}}\,dw}_{= \int_{C_\rho} f(w)(w-z_0)^{n-1}\,dw}$$

$= 0$ by Cauchy's when $n = 1, 2, 3, \ldots$ ✓

2) Poles. Lemma Laurent exansions are unique

Pf: Suppose $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$ (with unif convergence of $\sum_0^{\infty}$ and $\sum_{-\infty}^{-1}$ on $r < |z| < R$)

$$a_N = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{(z-z_0)^{N+1}}\,dz$$

$$= \frac{1}{2\pi i}\int_{C_\rho}\left[\sum_{-\infty}^{\infty} b_n (z-z_0)^n\right]\frac{1}{(z-z_0)^{N+1}}\,dz$$

$$\underset{\text{via uniform convergence}}{=} \sum_{n=-\infty}^{\infty}\left(\frac{b_n}{2\pi i}\underbrace{\int_{C_\rho}(z-z_0)^{n-N-1}\,dz}_{\substack{=0 \text{ if } n \neq N \\ = 2\pi i \text{ if } n = N.}}\right)$$

$$= b_N$$ ✓

<u>Pf of (2)</u> $(\Rightarrow)$ $f(z) = \underbrace{\left(\frac{a_{-N}}{(z-z_0)^N} + \dots + \frac{a_{-1}}{z-z_0}\right) + (\text{power series})}_{\text{Laurent expansion}}$

Easy case. See z_0 is a pole.

$(\Leftarrow)$ If z_0 is a pole, we know $f(z) = (\text{princ part}) + (\text{power series})$. Uniqueness of Laurent expansion $\Rightarrow$ this is <u>the</u> Laurent expansion.

3) Leftover case. Done ✓