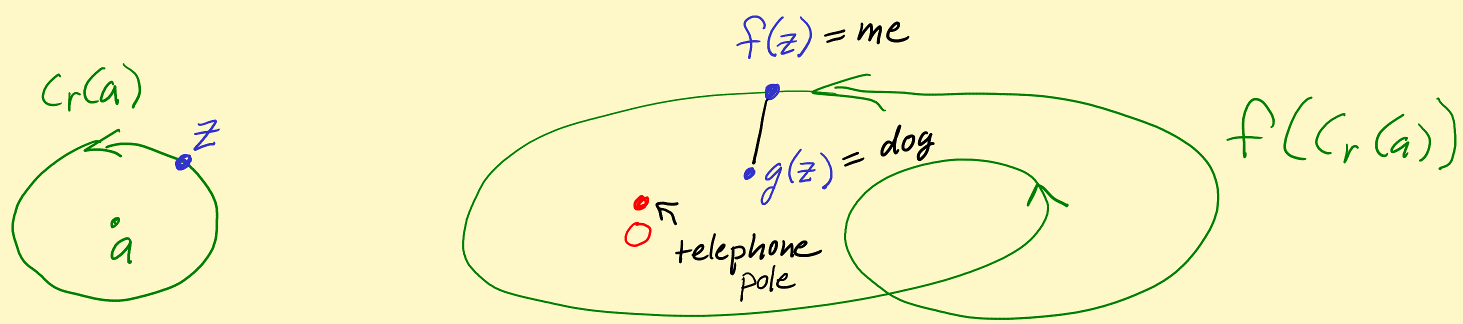
Lecture 29 Rouché's theorem & Hurwicz's theorems

Rouché's for a disc: Suppose f, g analytic on $D_R(a)$ and $0 < r < R$ and

$$(*) \quad |f(z) - g(z)| < |f(z)| \text{ on } C_r(a).$$

Then f & g have the same # of zeroes inside $C_r(a)$ (counted with multiplicity).

$$\left(\#\text{zeroes of } f \text{ inside } C_r(a)\right) = \left(\begin{array}{c}\#\text{ times } f(z) \text{ goes around } 0 \\ \text{as } z \text{ goes around } C_r(a)\end{array}\right)$$



$$\underbrace{|f(z) - g(z)|}_{\text{leash length}} < \underbrace{|f(z)|}_{\text{dist(me, pole)}}$$

Feeling: dog goes around pole same # times I do.
$= \left(\#\text{ zeroes of } g \text{ in } C_r(a)\right)$

Hurwicz's Thm # 1 Suppose f_n is a seq of analytic fcns on a domain Ω and $f_n \twoheadrightarrow f$ ($\twoheadrightarrow$ unif conv on compacts) on Ω. Then we know f is analytic (Morera's thm).

If all the f_n are non-vanishing on Ω, then either

A) $f \equiv 0$ on Ω, or

B) f is non-vanishing on Ω too.

Way false $\mathbb{R} \to \mathbb{R}$. $f_n(t) = t^2 + \frac{1}{n}$

EX $f_n(z) = z^n$ on $\Omega = D_1(0) - \{0\}$. $f_n \twoheadrightarrow 0$ on Ω.

Pf of H Suppose f_n non-vanishing, $\twoheadrightarrow f$.

Suppose $f(z_0) = 0$. If $f \not\equiv 0$, then z_0 is an isolated zero of f. So $\exists\ \overline{D_\varepsilon(z_0)} \subset \Omega$ so that z_0 is the only zero of f in $\overline{D_\varepsilon(z_0)}$. Let $m = \min\{|f(z)| : z \in C_\varepsilon(z_0)\} > 0$

Want to use Rouché's: $\underbrace{|f(z) - f_n(z)|}_{\substack{\text{unif on} \\ \text{compact set}}} < \underset{\substack{\uparrow \\ m=\varepsilon}}{m} \le |f(z)|$ on $\underset{\substack{\uparrow \\ \text{compact set}}}{C_\varepsilon(z_0)}$

So there is an N such we have this ineq when $n > N$.

Conclude: $\underset{\substack{\uparrow \\ \ge 1 \text{ zero}}}{f}$ and $\underset{\substack{\uparrow \\ \text{no zeroes!}}}{f_n}$ have same # zeroes in $C_\varepsilon(z_0)$, $n > N$. $\lightning$.

So f must be $\equiv 0$.

We've shown that either $f \equiv 0$, or it has no zeroes. ✓

Hurwicz's thm #2 Ω, $f_n \twoheadrightarrow f$, f_n analytic.

Suppose f_n are one-to-one on Ω. Then either

A) $f \equiv$ const on Ω, or

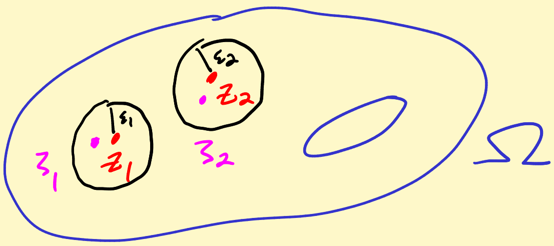
B) f is one-to-one on Ω too.

<u>Pf</u> Redo the #1 pf: Suppose f <u>not</u> 1-1. Then $\exists$

$z_1, z_2 \in \Omega$, $z_1 \neq z_2$ and w_0 such $\begin{cases} f(z_1) = w_0 \\ f(z_2) = w_0 \end{cases}$.

Then $f(z) - w_0$ has zeroes at z_1, z_2. If $f(z) \not\equiv w_0$,

then z_1 and z_2 are <u>isolated</u> zeroes of $f(z) - w_0$.

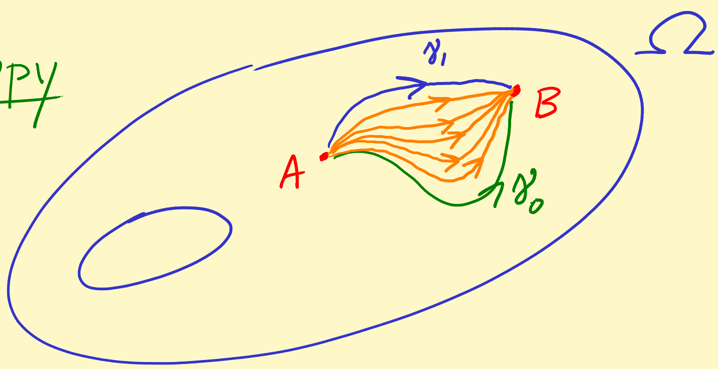


Shrink $\varepsilon_1, \varepsilon_2$ so that closed discs are disjoint in H #1 pf (using $f(z) - w_0$ in place of f).

$n > \mathrm{Max}(N_1, N_2)$: $\underbrace{\begin{pmatrix} \#\text{zeroes of} \\ f_n(z) - w_0 \\ \text{inside } C_{\varepsilon_j}(z_j) \end{pmatrix}}_{\text{conclude } \exists\, \zeta_j \in D_{\varepsilon_j}(z_j) \text{ with } f(\zeta_j) = w_0} = \underbrace{\begin{pmatrix} \#\text{ zeroes of} \\ f(z) - w_0 \\ \text{inside } C_{\varepsilon_j}(z_j) \end{pmatrix}}_{\geq 1}$

for $j = 1, 2$. f_n's not 1-1. $\lightning$. ✓

<u>Homotopy</u>



$\gamma_j : z_j(t)$, $0 \leq t \leq 1$

$\gamma_0 \sim \gamma_1$ in Ω $\quad (\gamma_0 \sim_\Omega \gamma_1)$: "γ_0 is homotopic to γ_1 in Ω"

means: there is a continuous fcn $H : [0,1] \times [0,1] \to \Omega$

such that $\quad H(0,t) = z_0(t)\ , \quad 0 \le t \le 1$

$$H(1,t) = z_1(t)\ , \quad 0 \le t \le 1$$

and $\quad \begin{cases} H(s,0) = A \\ H(s,1) = B \end{cases}$ for $\quad 0 \le s \le 1$

Think $\quad \gamma_s : z_s(t) = H(s,t) \leftarrow$ orange curves that stretch from γ_0 to γ_1 in Ω.

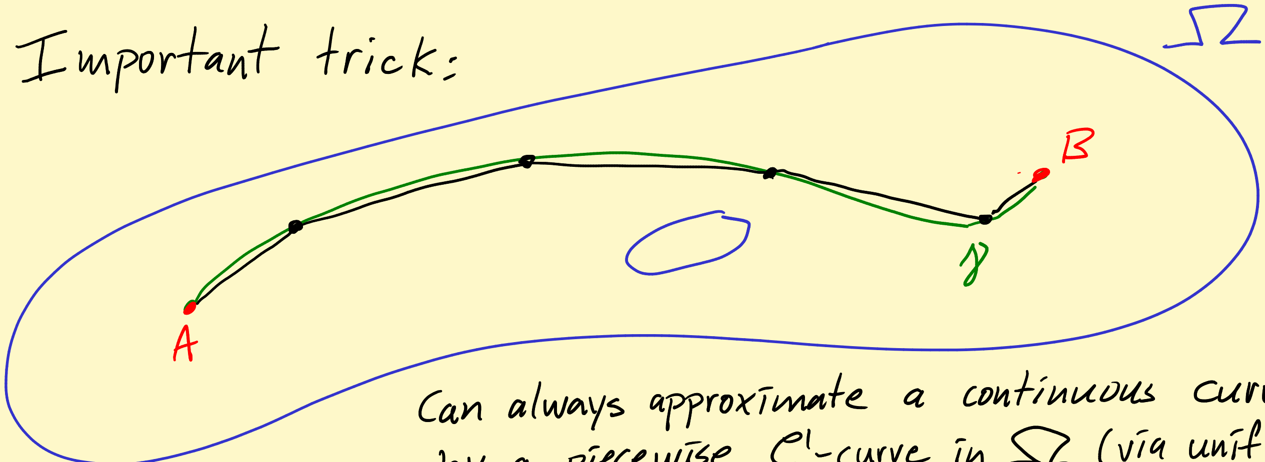
Big theorem If f analytic on a domain Ω and

$\gamma_0 \sim_\Omega \gamma_1$, then $\displaystyle\int_{\gamma_0} f\, dz = \int_{\gamma_1} f\, dz$, i.e.

$\displaystyle\int_{\gamma_A^B} f\, dz$ is independent of path.

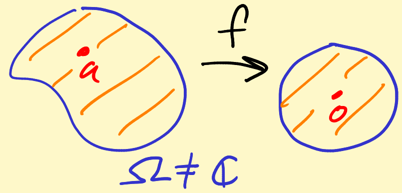
Hmmm. Ouch. We don't know what $\int_\gamma f\, dz$ means when γ is merely a continuous curve.

Important trick:



Can always approximate a continuous curve in Ω by a piecewise C^1-curve in Ω (via unif cont on a compact set.)

Sketch of pf of Riemann mapping thm.



$$\tilde{\mathcal{F}} : \left\{ h : h \text{ analytic on } \Omega,\ h : \Omega \xrightarrow[1\text{-}1]{} D_1(0) \right\}$$

HWK : Expect Riemann map to maximize $|f'(a)|$ among such maps.

(Schwarz!)

Strategy. Pick seq $h_n \in \tilde{\mathcal{F}}$ such $|h_n'(a)| \to M = \sup_{h \in \tilde{\mathcal{F}}} |h'(a)|$.

(Cauch estimate $\Rightarrow M < \infty$.)

Montel's thm $\Rightarrow \exists$ conv subseq $h_n \twoheadrightarrow h$.

Hurwicz #2 $\Rightarrow h$ 1-1 on Ω (since $|h'(a)| = M \neq 0$).

Use Schwarz again to see h is <u>onto</u> $D_1(0)$.