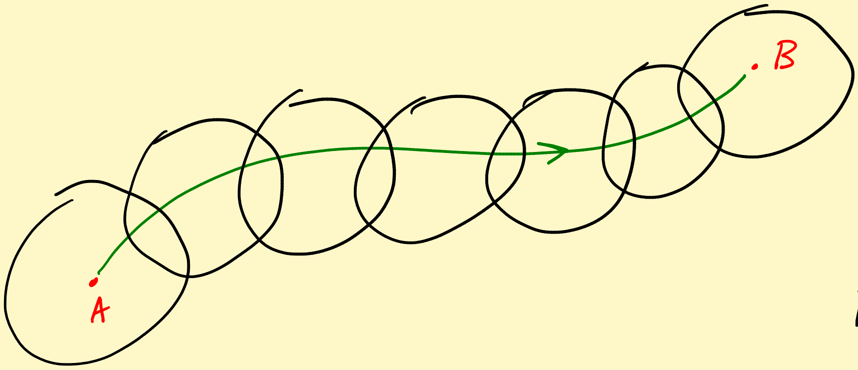
Lecture 31 Montel's theorem HWK 7 due tomorrow 11/7

Remark f analytic, γ merely a continuous curve: $\int_\gamma f\,dz$ can be defined



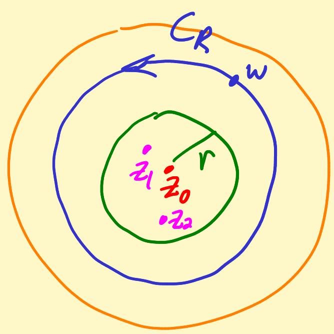
Ω "caterpilar" domain, simply connected.
$\int_A^B f\,dz$ for piecewise C^1 path indep. $\underset{\text{def}^n}{\overset{\uparrow}{=}} \int_\gamma f\,dz$

Arzela-Ascoli: Baby Rudin p. 158
Ahlfors p. 219 - 227
Stein p. 225

Big lemma A uniformly bounded family $\mathcal{F}$ of analytic fcns on a domain is uniformly equicontinuous on compact subsets.

Pf Cauchy integral formula

$f \in \mathcal{F}$



$D \subset \Omega$

$\max_{C_R} |f| < M$ ← indep of f

$$|f(z_1) - f(z_2)| = \left| \frac{1}{2\pi i} \int_{C_R} f(w) \underbrace{\left[\frac{1}{w-z_1} - \frac{1}{w-z_2} \right]}_{\frac{z_1-z_2}{(w-z_1)(w-z_2)}} dw \right|$$

$$\le |z_1 - z_2| \frac{1}{2\pi} \underbrace{\max_{C_R} |f|}_{\le M} \cdot \frac{1}{(R-r)^2} \; 2\pi R$$

$$|f(z_1) - f(z_2)| \le C\,|z_1 - z_2|, \quad C \text{ indep of } f \in \tilde{\mathcal{F}}.$$

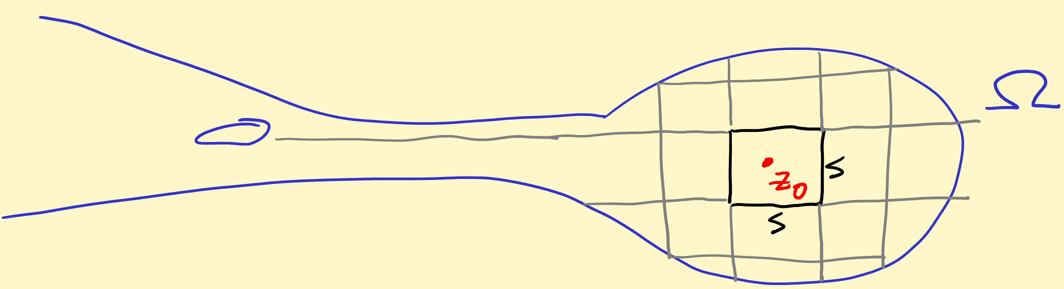
Finally, given compact set K, for each $z_0 \in K$, get a disc $D_r(z_0)$ as in picture. Let $\delta = \frac{r}{2}$. Get finite subcover.

Let $\delta =$ Min of deltas.

Montel's theorem Suppose f_n are analytic on a domain Ω and uniformly bounded on compact subsets of Ω. Then there is a subseq f_{n_k} that converges uniformly on compact subsets to f (which we know is analytic on Ω).

Pf Step 1 "Exhaust" Ω by compact sets =

$$K_1 \subset K_2 \subset K_3 \subset \cdots \qquad \bigcup_{n=1}^{\infty} K_n = \Omega$$



Start with square of side s compactly contained in Ω. Make grid, step s.

$$K_1 = \left[\text{Union of closed squares of side } s \subset \Omega\right] \cap D_{10s}(z_0)$$

$$K_2 = \left[\text{ '' '' '' side } \tfrac{s}{2}\right] \cap D_{100s}(z_0)$$

$$K_3 = \left[\text{ '' '' '' side } \tfrac{s}{2^2}\right] \cap D_{1000s}(z_0)$$

etc.

<u>Step 2</u> $z \in \mathbb{C}$: $\operatorname{Re} z$ and $\operatorname{Im} z$ in $\mathbb{Q}$ is a countable dense subset of $\mathbb{C}$. Fix K_j. Let $\{z_n\}_{n=1}^{\infty}$ be a countable dense subset of K_j from above dense set.

<u>Step 3</u> $f_1(z_1), f_2(z_1), f_3(z_1), \ldots$ is a <u>bndd</u> seq

So it has a convergent subseq

$f_{1,1}(z_1), f_{1,2}(z_1), f_{1,3}(z_1), \ldots \longrightarrow$ conv.

Now $f_{1,1}(z_2), f_{1,2}(z_2), f_{1,3}(z_2), \ldots$ is bndd.

So it has a conv subseq

$f_{2,1}(z_2), f_{2,2}(z_2), f_{2,3}(z_2), \ldots \longrightarrow$ conv.

Now $f_{2,1}(z_3), f_{2,2}(z_3), f_{2,3}(z_3), \ldots$ is bndd.

So it has a conv subseq

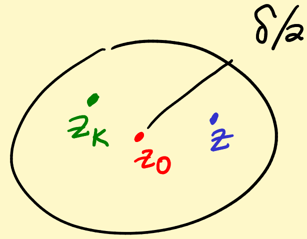
$f_{3,1}(z_3), f_{3,2}(z_3), f_{3,3}(z_3), \ldots \longrightarrow$ conv.

Aha! $\underbrace{f_{1,1}}_{F_1}, \underbrace{f_{2,2}}_{F_2}, \underbrace{f_{3,3}}_{F_3}, \underbrace{f_{4,4}}_{F_4}, \ldots$ is a subseq that conv at each z_n in dense subset!

<u>Step 4</u> Claim: F_n are <u>uniformly</u> Cauchy on K_j. Hence they converge uniformly on K_j.

Idea: $$|F_n(z) - F_m(z)| = |\underbrace{F_n(z) - F_n(z_k)}_{|\,| < \frac{\varepsilon}{3} \text{ by unif equi}} + \underbrace{F_n(z_k) - F_m(z_k)}_{|\,| < \frac{\varepsilon}{3} \text{ by convergence}} + \underbrace{F_m(z_k) - F_m(z)}_{|\,| < \frac{\varepsilon}{3} \text{ unif. equi}}|$$

How: Pick $z_0 \in K_j$. Get $\delta > 0$ for unif equicontinuity on K_j for $\frac{\varepsilon}{3}$ instead of ε.



Choose $z_K \in D_{\delta/2}(z_0)$

Aha! $z \in D_{\delta/2}(z_0)$. Then $|z_K - z| < \delta$ ✓

$\exists\, N$ such that $|F_n(z_K) - F_m(z_K)| < \frac{\varepsilon}{3}$ when $n, m > N$ because convergent seq are Cauchy.

Cover K_j by such discs. Take $N = \max$ of N's for discs.

<u>Step 5</u> Get subseq $F_{1,1}, F_{1,2}, F_{1,3}, \ldots$ conv unif on K_1.

Take subseq $F_{2,1}, F_{2,2}, F_{2,3}, \ldots$ conv unif on K_2.

Take subseq $F_{3,1}, F_{3,2}, F_{3,3} \ldots$ conv unif on K_3

$\vdots$

$F_{n,n}$ is a subseq that converges unif on each K_j!

<u>Step 6</u> Given any compact $K \subset \Omega$, it is eventually contained in a K_j. Get unif conv of $F_{n,n}$ on any compact.

Technical point: If Ω is a <u>bounded</u> domain: $|z| < M$ on Ω.

$$f(z) = \frac{z}{M} \text{ maps } \Omega \text{ 1-1 into } D_1(0).$$

What if $\Omega \neq \mathbb{C}$, but <u>not</u> bounded? Yes, 1-1 analytic $\to D_1(0)$ exist.